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1817

A BRIEF COURSE

IN

ELEMENTARY DYNAMICS

FOR

STUDENTS OF ENGINEERING

BY

ERVIN S. FERRY,
Professor of Physics in Purdue University

LAFAYETTE, INDIANA
1906

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PREFACE.

In preparing this text book it has been the aim to present a clear, consecutive, and consistent development of those laws and principles of dynamics which are of most frequent occurrence in the ordinary affairs of life and of widest application in the arts. In carrying out this purpose the following points have been kept in view:— (1), all technical terms should be accurately and succinctly defined; (2), the number of propositions derived from experience and experiment that are to be taken as fundamental should be as few as possible; (3), the laws of the subject should be deduced from the definitions and fundamental principles by rigid methods whenever the advancement of the student will justify it—when this is not feasible the weakness of the treatment should be indicated; (4), the laws should be enunciated in such terse and inclusive statements that they can be easily assimilated and readily applied; (5), the laws should be vivified in the mind of the student by numerous illustrations drawn from applications with which he is familiar and in which he is interested; (6), since mistakes are due not only to failure to comprehend laws but also to faulty reasoning from them, the student should be trained in solving numerical problems. For engineers, especially, is facility in solving numerical physical problems of enormous importance.

It will be noticed that the body of the text is in large type, while problems, examples, and all illustrative material has been put into smaller type. This arrangement assists in obtaining a proper perspective of the various parts, and in a review permits the "story" to be followed without the attention being distracted by unessentials. A few articles containing long proofs, which might with propriety be omitted in the first reading, have been marked with an asterisk. In all cases, however, the conclusion printed in italics should be learned.

The problems at the ends of various sections have been selected out of a card index which has been collected from various sources through a number of years. Few of them are original, but the sources from which they were obtained have been lost.

The author is under deep obligation to his colleague Assistant Professor Arthur T. Jones who has read the entire proof and has pointed out several errors and rendered clear many obscure passages. To Professor Jones is also due the entire section on wave motion.

ERVIN S. FERRY.

LaFayette, Indiana, Sept. 15, 1906.

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CHAPTER I.

FUNDAMENTAL NOTIONS.

§ 1. *Force.*

1. **Stress and Force.**—Two bodies at the ends of a compressed spiral spring tend to recede from one another. If a magnet and a piece of iron, both free to move, be placed near one another, each will move toward the other. The distance between the earth and any body near it always tends to diminish. When two bodies interact so as either to produce or tend to produce motion, their mutual action is called a *stress.* When the attention is confined to the cause which tends to give motion to one of the bodies, the other body being left out of consideration, the action of the stress on this body is called force. *Force* may be defined as any cause which either changes or tends to change the motion of a body relative to another. If the attention be limited to the motion of one of the bodies, the force acting is said to be an *external* or *impressed* force. If, on the other hand, the two bodies be considered as forming a single system, and no force from outside the system acts upon it, the force acting upon each body is called an *internal* force. Other names sometimes used for force are thrust, push, pull, pressure, tension, attraction, repulsion, effort, resistance.

Whenever there is a stress between two bodies there must be a medium extending from one to the other. In some cases this intervening medium is not obvious. In the first illustration, above, the effect is due to a stress in the spiral spring between the two bodies. In the case of the magnet and the piece of iron the medium is an invisible, all pervading, highly attenuated substance whose properties have been pretty thoroughly studied. It is known that this medium, called the *ether*, serves for the propagation of light, electric, and magnectic effects, and it is highly probable that gravitational attraction is also due to a stress of this medium.

2. Reaction.—Imagine two boys pulling in opposite directions on a rope which passes through a hole in a fence. If the two boys are equally strong, the rope will not move in either direction. If one boy should tie his end of the rope to a post, the other would be quite unaware of the fact. That is, the post offers the same resistance to the force exerted by the boy pulling the rope as did the other boy. In other words, the force exerted on the post develops an equal and oppositely directed reaction on the boy pulling the rope.

A stress always has two ends—it always exerts a mutual action upon two different bodies or upon two parts of a single body. The two ends of a stress are called its *action* or *force*, and its *reaction*. Either end may be called the action or force, and then the other end would be called the reaction of the force. The reaction of a force is developed by the force and exists just so long as the force acts. It should be carefully noted that *a force and its reaction always act upon different bodies.*

This two-endedness of a force is described by the statement that *every force has an equal and oppositely directed reaction.* This is called Newton's Third Law of Motion.

3. Representation of Forces.—For the complete specification of a force three characteristics are necessary. They are, *magnitude, direction,* and *point of application.* Since a straight line is also completely described by these same characteristics, a force can be represented by a straight line. For example, in the case of a body resting on a table as shown in Fig. 1, the weight of the body, *i. e.*, the force acting downward on the table, is represented in direction and point of application by the line *CA*. If, in addition, the length of *CA* is proportional to the magnitude of the force exerted on the table by the body, then the line *CA* represents completely the force acting on the table. In the same manner, the reaction exerted by the table on the body is represented by the line *BA*.

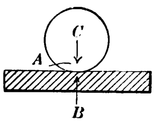

Fig. 1.

4. Illustrations of Force and Reaction.—If in the case of a body falling toward the earth, (Fig. 2), we speak of the force of gravity, F_1 acting on the body, then the reaction would be an equal and oppositely directed force, F_1', acting on the earth.

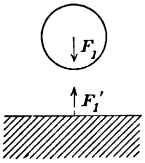

Fig. 2.

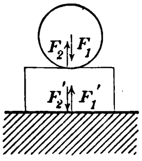

Fig. 3.

If instead of falling, the body rests upon a table, there are the same action and reaction due to the earth's gravitational force as before (F_1 and F_1', Fig. 3). But there is now an additional stress due to the passive resistance offered by the table which exactly counteracts the tendency of the body and the earth to come together. The table pushes upward on the body with a force F_2 while the table pushes downward upon the earth with an equal and oppositely directed force F_2'.

It must not be forgotten that the action and the reaction are always applied to different bodies. If two forces are applied to the same body so as to neutralize one another's effect, one is called the action and the other is called the *counteraction*. Action and reaction are the two ends of a single stress: action and counteraction are two ends of different stresses.

5.—Consider the case of the horse and cart. One often hears the query, "If a horse pulls a cart and the cart pulls backward on the horse to an equal extent, how can either the horse or cart move with reference to the earth?"

In Fig. 4 let the rectangle *A* represent the horse and the rectangle *B*, the cart. The horse's feet push backward against the earth with a certain force F_1, while the earth pushes against the horse's feet in the opposite direction with equal force. The horse pulls on the cart with a force F_2, while the cart pulls backward on the horse with equal force. Frictional and other passive resistances between the cart and the earth exert on the earth a force F_3 which develops an equal and oppositely directed reaction on the cart. Call forces acting in the direction of motion positive, and forces in the opposite direction negative.

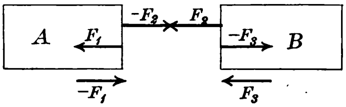

Fig. 4.

Then, if the road is level, the total force acting on the horse is $(F_1 - F_2)$, and the total force acting on the cart is $(F_2 - F_3)$. If the motion of the horse and cart is uniform, the total force acting on the horse is

$$(F_1 - F_2) = 0,$$

and the total force acting on the cart is

$$(F_2 - F_3) = 0.$$

Therefore, in order to keep the cart moving uniformly, the horse needs to exert a force

$$F_1 = F_2 = F_3.$$

If, however, the horse is to increase the cart's motion—for example, if he is to start it from rest,—then both $(F_1 - F_2)$ and $(F_2 - F_3)$ must be greater than zero. That is, we must have F_1 greater than F_2, and F_2 greater than F_3.

6. The Effects of Force.—The primary effects of force are two in number. A force can cause distortion, that is, a displacement of one part of a body relative to another part. Or, a force can cause a change in the motion of a body relative to its surroundings. This change of motion may be either a change of direction or a change of magnitude. By the motion of a body is meant the change of its position with reference to some other body.

A motion which changes either in magnitude or direction is said to be *accelerated*. The rate of change of the motion of a body, that is, the change of motion per unit of time is called *acceleration.*

7. Equilibrium defined.—A system of forces applied to a body is in *equilibrium* if the motion of the body is unchanged by the action of the forces. A body is said to be in equilibrium when the forces acting upon it are in equilibrium. If the system of forces acting upon a body is in equilibrium, the combined effect of all the forces on the motion of the body is zero.

8. Measure of a Force.—Measurement consists in the comparison of the magnitude of the assigned quantity with the magnitude of an arbitrarily selected unit taken as standard. When the quantity to be measured is of a nature such that the unit can take the form of a material standard, the measurement can be made *directly*. Thus a distance can be measured directly by means of a bar of unit length. It is either impossible or inconvenient, however, to measure directly many physical quantities. For instance, two temperatures cannot be directly compared. But since the volume of a body depends upon its temperature, and volumes can be directly measured, temperatures can be *indirectly* compared. A physical quantity can be measured indirectly in terms of any directly measurable effect it is capable of producing.

The effects of force are two. Force can produce (1) a change of the shape or size of a body, or (2) a change in the motion of the body. Since each of these effects is directly measurable, each furnishes a method of measuring force. Thus, the unit of force might be defined as that force which can cause a given body to be distorted to a given extent, or, as that force which can cause a given change in the speed of a body in a given time. Both of these methods of measuring force are employed—the first in engineering and in practical life, and the second in scientific work.

9. The Gravitational Units of Force.—One of the facts first discovered in the history of science is that any body on the surface of the earth exerts a force toward the earth. This force is called *weight* or the *force of gravity*. The weights of bodies at a given place can be compared by means of the distortions they produce in a spiral spring when suspended freely under the influence of the earth's attraction. For instance, if each of two bodies *A* and *B* when suspended from the end of a vertical spiral spring produces the same elongation of the spring, they are of equal weight. If a third body produces the same elongation as do *A* and *B* together, then the weight of the third body equals twice the weight of *A* or *B*. Proceeding in this manner, a spiral spring can be "calibrated" so as to give directly the weights of bodies, compared to any unit weight taken as a standard of comparison. This is the essence of the earliest and simplest method of comparing forces. It is a method still used in engineering and in ordinary life.

In English speaking countries, the gravitational unit of force is the weight at London of a certain piece of platinum preserved in the Office of the Exchequer. It is called the *pound weight avoirdupois.* On the continent of Europe the gravitational unit of force is taken as the weight at Paris of a certain lump of platinum deposited in the Archives of Paris. This unit of force is called the *kilogram weight.*

1 kilogram weight = 2.2046 pounds' weight.

1 pound weight = 0.45359 kilogram weight.

The attraction exerted by the earth on a given body depends upon the latitude of the place at which the body is situated, upon its distance from the center of the earth, and upon such local conditions as the proximity of great mountains, etc. Consequently, the weight of a body is an indefinite quantity depending upon the position of the body. At sea level, the ratio of the weight of a given body situated at the equator to its weight at the pole is as 978 is to 983. It is useless to take account of this small difference in engineering work, but in scientific work a unit of force must be employed which is absolutely constant. The absolute unit of force will be defined in a later paragraph.

In order to familiarize the student with the use of both gravitational and absolute units of forces, both will be used in the problems on succeeding pages.

10. Inertia, and Newton's First Law of Motion.—A slight blow with the fist will impart to an ordinary punching bag a considerable speed; but if the bag were filled with sand a much stronger blow would be required to give it the same speed. Little force is needed to stop a moving foot ball, but a considerable force would be required to stop a cannon ball moving with the same speed. A small force is sufficient to deflect from a straight path a stream of water issuing from a garden hose, but to cause a railway train to go around a curve requires well spiked rails. That characteristic of matter that requires force to change either the magnitude or direction of its motion is called *inertia.* Inertia is the distinguishing characteristic of matter. The magnitude of the intertia of a body is measured by the tendency of the body to continue to move in a straight line with constant linear speed.

This great Law of Inertia, or Newton's First Law, has been enunciated in the form, *a body will continue to move with its present speed, in a straight line, until acted upon by an external force.*

Illustrations abound of the effect of inertia. If a railway train suddenly stops, the passengers will be thrown forward on account of the inertia of their bodies preventing them from instantly changing their motion to that of the train. The sudden blow produced by the explosion of gun powder in a rifle will upset the end of the bullet so that it is compelled to follow the groove in the rifle barrel. Water pipes are often burst due to the continuance of the motion of the water after the valve has been closed. To guard against this accident, the large water gates in city water mains are arranged so that it is impossible to close them quickly. Locomotives are supplied with water while in motion by the simple device of dropping the forward end of an inclined pipe into a long trough of water extending parallel to the tracks. As the water in the trough tends to remain at rest with reference to the ground, the moving inclined pipe slips under a portion of the water and raises it vertically until it falls into the water tank on the tender.

11. Newton's Second Law of Motion.—Imagine that we have a number of spiral springs such that if they be acted upon by forces of 1, 2, 3, etc., pounds' weight respectively, they will be compressed by the same amount. If these springs while compressed by this amount be applied in turn to a given body, and while in contact be allowed to expand, the body will be given accelerations in the direction of the applied forces in the ratio of 1 : 2 : 3 : etc. In other words, *when a force acts upon a body, there is produced in the body's motion an acceleration whose direction is that of the force, and whose magnitude is directly proportional to that of the force applied.*

This law has been thoroughly tested and verified in the most diverse ways and is taken as one of the fundamental principles of dynamics. It is usually called Newton's Second Law of Motion.

12. The Principle of the Independence of Forces.—As the result of experience and of observations extending through several centuries, it has been concluded that, *if several forces act simultaneously upon a body each force produces its own effect independently of all the others.* This is called the Principle of the Independence of Forces.

13. The Principle of the Transmissibility of Forces.—It is a matter of common experience that the motion of a body is unchanged by the application of two equal and oppositely directed forces acting in the same straight line. Thus, if F_1 and F_2 are two equal and oppositely directed forces acting in the same straight line, they will produce no change in the motion of the body to which they are applied. But the force F_1 acting at A can also be balanced by an equal and oppositely directed force F_3 acting at the same point. Thus each of the two equal forces F_3 and F_2 acting in the same direction at the points A and B respectively can be balanced by the same force F_1. That is, the force F_3 acting at A produces the same effect as an equal, similarly directed force F_2 acting at B.

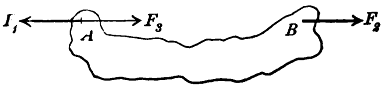

Fig. 5.

From this we conclude that *when a force acts upon a rigid body its effect is the same whatever point of its line of action is taken as the point of application.* This is called the Principle of the Transmissibility of Forces.

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§ 2. *Units of Length.*

14. The Foot.—By legislative enactment the various countries have legalized certain standards of length. There are but two that are extensively used. The *foot* is one third of the British Standard Yard. This latter is by law defined to be the distance, at the temperature 62° F., between the centers of two transverse lines on a certain bronze bar deposited in the Office of the Exchequer in London. The foot is the unit of length employed in engineering and in ordinary life throughout all English speaking countries.

15. The Centimeter.—The unit of length in common use throughout the continent of Europe is called the *meter*. The meter is the distance, at the temperature of melting ice, between the ends of a certain platinum bar preserved in the Archives in Paris. One thousand meters is called a kilometer; one tenth of the meter is called a decimeter; one hundredth is called a centimeter; and one thousandth is called a millimeter. In scientific work, throughout the world, the centimeter is used as the unit of length.

The meter was intended to be one ten millionth part of the distance from the equator to one of the poles of the earth. In 1799 very careful surveys were made for the purpose of determining this distance, and from this measurement the meter was constructed. More recent surveys have shown this measurement to be slightly in error. It is now known that a quadrant of the earth's surface instead of measuring 10,000,000 meters, is more nearly equal to 10,000,880 meters.

The relation between the two units of length is shown in the following table:

1 centimeter	= 0.39371 inch	1 inch	= 2.5399 centimeters
1 meter	= 3.2809 feet	1 foot	= 0.30479 meter
1 kilometer	= 0.62139 mile	1 mile	= 1.6093 kilometers.

§ 3. *Work and Energy.*

16. Work defined.—When, in opposition to a resisting force, the motion of a body is changed, work is said to be done against the force. *Work* may be defined as the accomplishment of a change in the position of a body against an opposing force.

If a body is lifted, work is done against the force of gravity. If a spiral spring is compressed, work is done against the force of the spring. If a body is dragged along the ground, work is done against the force of friction. If a man swims up-stream he does work against the current whether he is advancing or whether his position with reference to the earth does not change. Since in walking a man raises himself at each step, work is here done against the force of gravity. Since our muscles are constantly relaxing and recovering, work is done against the force of gravity when we attempt to stand erect for any considerable length of time, or when we hold a heavy body with outstretched arm.

Walking consists of a series of intercepted falls. At each step the body is raised against the force of gravity through a certain distance, and then allowed to fall through the same distance. This distance is about one and one fourth inches for a man, and about three inches for a horse. Thus at each step a positive amount of work is done against the force of gravity, succeeded by an equal negative amount. The total work done is zero. The sensation of fatigue experienced depends upon the positive work done.

No work is done when a body rests upon a table or when the pressure of the steam in an engine does not move the piston. No work is done when a piece of iron is supported either by a string or by a magnet.

It should be kept in mind that *for the performance of work, the two bodies between which the stress exists must move with respect to one another.*

17. Energy defined.—If a spiral spring be compressed, a force must be exerted through a certain distance—that is, work must be done on the spring. If the compressed spring be released it can, in turn, do work on some other body. Similarly, if a body be raised from the earth against the force of gravity, work must be done upon the body. If the elevated body be allowed to fall, it can do work on some other body. Again, by applying force at the shaft of a fly-wheel it can be set into rotation. After being set into rotation the fly-wheel can do work on some other body.

In the above examples of the distorted spring, the body raised against gravity, and the body set into rotation, all have the ability to do work on account of work having previously been done upon them in bringing them into their present condition. The ability to do work is called *energy*. Energy is stored work. The quantity of energy possessed by a system of bodies is the amount of work it can do against external forces in passing from its present condition to some standard condition. Energy is capable of transformation into many forms. Thus we have mechanical energy, heat energy, electrical energy, etc. Energy in any form can be transformed into any other form.

18. The Principle of the Conservation of Energy.—Whenever work is done, one system of bodies loses energy and another gains energy. The results of most carefully planned and accurately executed experiments indicate that *the amount of energy lost by one system of bodies always equals the amount of energy gained by another.* In other words energy can neither be created nor destroyed. All observed natural phenomena are in accord with this conclusion. This very important generalization is called the Principle of the Conservation of Energy.

It should be noted that there is no such thing as a law of the conservation of *force.* Force can be either created or destroyed at one part of a system of bodies without any corresponding change of force taking place in the remainder of the system. Force may appear or disappear without any change in the energy of the system. Work is not required to produce force nor is energy necessarily liberated when force disappears. The failure to appreciate these facts has been a fruitful source of disappointment for inventors of ''perpetual motion'' machines.

19. *Measure of Work.—From the definition, (Art. 16), the quantity of work done against an opposing force depends upon both the magnitude of this force and the displacement effected against this force. As an arbitrary convention, if a body is moved in the line of action of an opposing force, the magnitude of the work done, for a given change of the distance between the bodies between which the stress exists, is taken to be directly proportional to the resisting force; and, for a given resisting force, the work done is taken to be directly proportional to the change of distance between the bodies between which the stress exists.

A combination of these two variations gives the law (1), *if a body is moved in the direction of the line of action of a force, the work done is measured by the product of the force and the displacement of the body.*

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(1) "If A varies as B when C is constant, and A varies as C when B is constant, then will A vary as BC when both B and C vary."—Hall & Knight, Higher Algebra, p. 23.

Work done against a force is called positive and work done by a force is called negative.

The value of the amount of work done when the displacement of the body is perpendicular to the line of action of the impressed force will now be determined. If, while a force F acts upon it, a body M is moved in the direction from A toward B, (Fig. 6), then the work done, unless it is zero, must be either positive or negative. If the body were moved in the opposite direction the work would be of the opposite sign. If Fig. 6 be viewed from the back of the sheet on which it is printed its appearance would be essentially that of Fig. 7. Since the sign of the work is independent of the position of the observer, if in Fig. 6 when the body is moved from A toward B the work be called positive, then in Fig. 7 when the body is moved from A toward B the work is also positive. If however, the direction of motion be reversed, (Fig. 8), the work would be negative. Comparing Figs. 6 and 8 it is seen that for a displacement of the body in a given direction, (for example, from left to right along the line AB), the work changes sign when the angle between the line of displacement of the body and the line of action of the force changes from $\theta°$ to $(180° - \theta)$. Consequently, when $\theta = 90°$ the work is zero. In other words, *if a body moves perpendicularly to the line of action of the impressed force no work is done either by the force or against the force.*

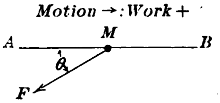

Fig. 6.

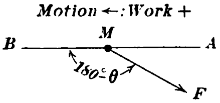

Fig. 7.

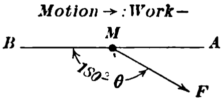

Fig. 8.

From the two laws above deduced, together with the principle of the conservation of energy, we can now derive the value of work done when a body is displaced in a direction inclined to the line of action of the impressed force. To fix the ideas, consider a body resting on a smooth inclined plane to be under the action of the force of gravity. Imagine that the body be allowed to move down the inclined

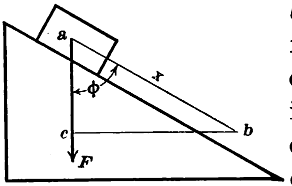

Fig. 9.

plane from a to b through a distance x, then horizontally from b to c, and then be lifted vertically from c and a. Let W' be the work done on the body by the force of gravity during the displacement ab. Since the displacement bc is normal to the line of action of the impressed force, the work done during this displacement is zero. Since ca is in the direction of the impressed force, the work done on the body during this displacement against the force of gravity F is

$$W = F\ (ca).$$

Whence, in traversing the cycle $abca$, the total work done *by* the force equals $(W' - W)$. The magnitude of W relative to W' is now to be determined. If it be assumed that W' is greater than W, then every time the cycle $abca$ is traversed a positive amount of energy will be created. Since this result is quite contrary to the principle of the conservation of energy, the hypothesis that W' is greater than W must be untrue. Similarly, it can be shown that W' is not less than W. Whence $W' = W$. That is, if a body is displaced through a distance x in a direction making an angle Φ with the line of action of a force F, the work done by (or against) the force is

$$W = Fx \cos \Phi. \tag{1}$$

Or, in words, *the work done by (or against) a force equals the product of the force and the projection, on its line of action, of the displacement of the body.*

When distances are measured in feet and forces in pounds' weight, the unit of work is called the *foot-pound*. When distances are measured in meters and forces in kilograms' weight, the unit of work is called the *kilogram-meter*.

20. The Work Principle.—A useful corollary to the principle of the conservation of energy can be enunciated in the following form: *When a system of forces in equilibrium is applied to a body and a small displacement produced, the work of the acting forces is equal to the work of the resisting forces.* This is called the Work Principle.

§ 4. *General Considerations.*

21. Dynamics defined.—Any great department of knowledge in which the results of investigation have been correlated and systematized is called a *science*. *Physics* is the science of energy. It investigates those phenomena of nature which depend either upon the transfer of energy from one body to another, or upon the transformation of energy from one form into another. For convenience of study, physics is frequently divided into the following branches: Dynamics, Heat, Sound, Light, Electricity and Magnetism. *Dynamics* is that branch of physics which investigates force. Dynamics is divided into Statics and Kinetics. *Statics* treats of forces in equilibrium, while *kinetics* treats of the relation of force to motion.

Newton gave the name *mechanics* to the science of machines or the application of dynamics to machinery and engineering structures. In this he is followed by most logical modern writers although it is not infrequent to see the terms dynamics and mechanics used interchangeably.

22. The Basis of dynamics.—Dynamics is the oldest, most far reaching, and most thoroughly systemetized branch of physics. In addition to being the foundation of physics its laws constitute an important part of all the other natural sciences.

Dynamics is based upon certain arbitrary definitions, and certain fundamental principles or grand generalizations obtained from experience and experiment. Inasmuch as these principles cannot be derived from one another or from any simpler laws, and since all the laws of dynamics can be deduced from them, they constitute the basis of dynamics. These fundamental principles are five in number. They are

1.—Newton's First Law of Motion.
2.—Newton's Second Law of Motion.
3.—The Principle of the Independence of Forces.
4.—The Principle of the Transmissibility of Forces.
5.—The Principle of the Conservation of Energy.

STATICS OR THE LAWS OF EQUILIBRIUM.

CHAPTER II.

Composition and Resolution of Forces.

§ 1. *Statical Moments.*

23. Translation and Rotation defined.—By *translation* is meant a displacement of a body from one position to another in such a manner that all points of the body traverse equal, parallel paths. A displacement of a body such that all points describe circular paths about the same axis is called *rotation.* The axis of rotation may be outside the body or it may pass through the body.

Motion of translation is often called *linear motion,* and motion of rotation is often called *angular motion.*

Motions in opposite directions are distinguished by the signs plus and minus. Rotation in the direction opposite to the motion of the hands of a clock is usually termed positive, while rotation in the direction of the hands of a clock is termed negative.

24. *Moments of Forces.—The *moment* of any physical agency is the numerical measure of the importance of its effect on rotation.

Consider a rigid rod AB acted upon by a force F_1 tending to produce a rotation in the positive direction about an axis normal to the paper through some point C. Let F_2 be a force in the same plane as F_1 which is resisting the action of F_1, and let F_3 be the pressure of the fulcrum at C against the rod . Let the rod rotate into a new position $A'B'$ through an angle so small that the arc described by A can be taken equal to the chord AA'. Let s_1 and s_2 be the linear displacements of the points of application of the forces F_1 and F_2 respectively, and let h_1 and h_2 be the projections of these displacements on the lines of action of the forces.

During this infinitesimal rotation the force F_1 will do an amount of work equal to $F_1\ h_1$, (Art. 19), while there will be

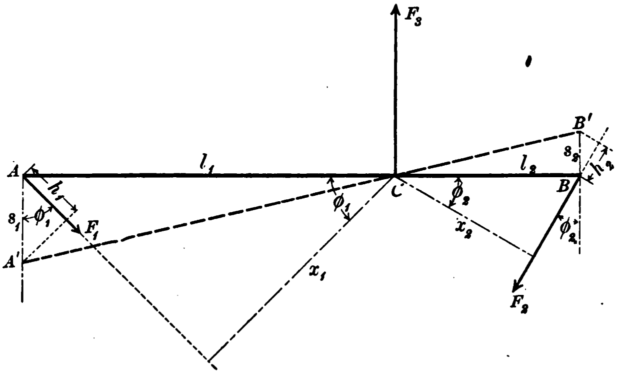

Fig 10.

done against the force F_2 an amount of work equal to $F_2\ h_2$. Since the fulcrum at C does not move, no work is done by or against the force F_3. If the rotation of the rod is sufficiently slow, the change in its motion may be neglected. In this case there will be no resistance due to inertia (Art. 10), consequently, from the "work principle" (Art. 20),

$$F_1\ h_1 = F_2\ h_2. \tag{2}$$

Denote by Φ_1 the angle between the line of action of the force F_1 and the direction of the displacement of its point of application. Similarly, let Φ_2 be the angle between F_2 and s_2. Then,

$$h_1 = s_1 \cos \Phi_1 \quad \text{and} \quad h_2 = s_2 \cos \Phi_2\ . \tag{3}$$

Denoting by l_1 and l_2 the distances AC and BC respectively, and denoting by x_1 and x_2 the normal distances of C from the lines of action of the two forces, we have

$$\cos \Phi_1 = \frac{x_1}{l_1} \quad \text{and} \quad \cos \Phi_2 = \frac{x_2}{l_2}. \tag{4}$$

Substituting in (2) the values obtained from (3) and (4), we obtain

$$\frac{F_1\ s_1\ x_1}{l_1} = \frac{F_2\ s_2\ x_2}{l_2}.$$

But from the similar triangles $AA'C$ and $BB'C$ we have

$$\frac{s_1}{l_1} = \frac{s_2}{l_2}.$$

Therefore

$$F_1\, x_1 = F_2\, x_2. \tag{5}$$

It should be noted that this equation was deduced on the supposition that the motion of the rod was approximately constant: in other words, this equation is strictly true only when there is no change in the motion of the rod, *i. e.* when the rod is in equilibrium.

The normal distance from the line of action of a force to the axis about which rotation is supposed to occur is called the *lever arm* of the force. Equation (5) shows that if the rotating effect of one force acting upon a body is balanced by the rotating effect of another force acting upon the body, so that the body is in equilibrium, then the product of either force by its lever arm equals the product of the other force by its lever arm. Now if in place of the force F_1 there were applied at the same point and in the same direction a force nF_1, the effect in producing or preventing rotation would be n times as great as before. But in order to balance this nF_1 the product of F_2 and its lever arm would need to be n times as great as before. The value of the product of a force and its lever arm may, therefore, be taken as a measure of the importance of the force upon the rotation of the body.

The importance of a force upon the rotation of a body is termed the *moment* of the force. A moment tending to produce rotation about the given axis in the clockwise direction is termed *negative*, one tending to produce rotation in the counter-clockwise directon is termed *positive*. The above equation can now be stated in the following concise form: *When a body is in equilibrium under the action of three forces, the algebraic sum of the moments of two of them about the point of application of the third, equals zero.*

When distances are measured in feet and forces in pounds' weight, moments of forces are expressed in *pound-feet.* When distances are measured in meters and forces in kilograms' weight, moments of forces are expressed in *meter-kilograms.*

25. The above method of proof may also be applied to show that *if a body is in equilibrium under the action of any number of co-planar forces, the algebraic sum of the moments of all the forces, about any axis perpendicular to their plane, equals zero.*

This theorem is expressed analytically by the equation

$$[F_1 x_1 + F_2 x_2 + F_3 x_3 + \dots\dots \text{etc.}] = 0, \qquad (6)$$

or by the abbreviated form

$$\Sigma\ (F\ x) = 0, \qquad (7)$$

where the symbol $\Sigma\ (F\ x)$ signifies, "the sum of a series of such terms as (Fx)".

In (6) and (7) due regard must be paid to the sign of each term. This law is called the Theorem of Moments.

Calling the single force that is equivalent to two or more forces the *resultant* of the set, a useful corollary to this theorem can be stated in the following form: *The moment of the resultant of a system of coplanar forces, about any axis perpendicular to their plane, equals the algebraic sum of the moments of the forces about that axis.*

26. Illustrations of Moments of Forces.—In Fig. 11, let *c* be the point of contact of the edge of a spool of thread with the table. The loose end of the thread is pulled with a force *F*. This force tends to make the spool turn in one direction or the other about the point *c*. If the line *cd* is above the surface of the table, then the moment of *F* about *c* is a negative moment and the spool rolls to the right; if, however, the line *cd* falls below the top of the table, the moment of *F* about *c* is positive and the spool rolls away to the left.

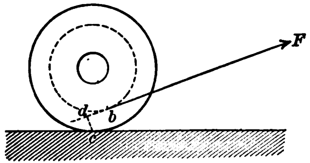

Fig. 11.

27. The lever is a familiar application of the theorem of moments. It consists of a rigid rod either straight or bent, capable of rotation about a fixed axis called the fulcrum. The forces applied to a lever are in a plane perpendicular to this axis. If there are but two forces, the position of the fulcrum with reference to these forces gives rise to three classes of levers which can be studied in the following three diagrams.

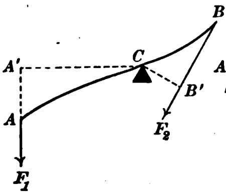

Fig. 12.

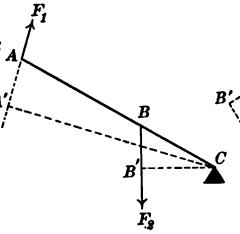

Fig. 13.

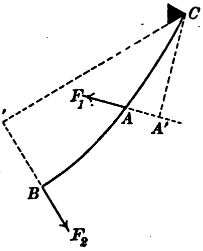

Fig. 14.

Let C be the fulcrum about which rotation can take place; and let F_1 and F_2 be two forces in equilibrium. The condition of equilibrium is now to be determined.

The same analysis applies to all three cases. From C draw lines CA' and CB' perpendicular to the lines of action of the forces. Then since the lever is in equilibrium, the sum of the moments of the forces about the fulcrum must equal zero. That is,

$$F_1\ (CA') - F_2\ (CB') = 0$$

In the special case where the lever is straight and the lines of action of both the forces are normal to it, we have

$$F_1\ (CA) = F_2\ (CB).$$

PROBLEMS.

For the present it may be assumed that the weight of a uniform body of regular shape is applied at its geometrical center. A "light" body is one of negligible weight.

1.—A uniform rod 12 ft. long and weighing 24 lb. rests horizontally on two props distant 2 ft. and 4 ft. from the two ends. Find the force supported by each prop. [8 lb. wt.; 16 lb. wt.]

2.—A uniform beam 12 ft. long and weighing 100 lb. rests horizontally on similar supports at its ends. Find the force supported by each prop when a load of 70 lb. wt. is placed one third of the distance from one end. [96.66 lb. wt.; 73.33 lb. wt.]

3.—A uniform iron rail weighing 100 lb. is supported by two posts 10 ft. apart, the posts being 12 ft. and 8 ft., respectively, from the ends of the rail. Find the weight each post must bear. [70 lb. wt.; 30 lb. wt.]

4.—A stiff pole 12 ft. long sticks out horizontally from a vertical wall. It would break if 28 lb. wt. were applied normally at the end. How far out along the pole may a boy safely venture who weighs 112 lb.? [3 ft.]

5.—A horizontal uniform cylinder, of which a portion 6 in. long weighs 8 lb., is 4.5 ft. in length. A body weighing 18 lb. being attached to one of its extremities, find the position of the fulcrum upon which the whole will balance. [21.6 in. from loaded end.]

6.—Two boys make a see-saw by balancing on a fence a uniform board 5 m. long. On one end sits a boy weighing 35 kg. Where must the other boy weighing 40 kg. place himself in order to balance the first boy? [31.25 cm. from the end.]

7.—Two men carry a body weighing 90 kg. suspended from a light pole 3 m. long. If the body be placed at a distance of 1.2 m. from one end, what weight does each man bear? [54 kg.wt.; 36 kg. wt.]

8.—A uniform plank 10 ft. long having a body weighing 25 lb. fastened to one end is balanced at a point 3 ft. from the loaded end. Find the weight of the plank. [37.5 lb. wt.]

9.—A beam of uniform cross section is carried by three men, one at one end and two by means of a light bar placed crosswise under the beam. Where must the bar be placed that each man may bear one third the weight? [¼ distance from end of beam.]

10.—A man carries on his shoulder a uniform straight pole weighing 2 kg., on one end of which hangs a body weighing 10 kg. He keeps the pole horizontal by holding down the other end with his hand. If the distance from shoulder to hand is 70 cm. and the distance from shoulder to suspended body is 2 m. find the vertical force exerted by the hand, and the weight on the shoulder. [30.4 kg. wt.; 42.4 kg. wt.]

11.—Find the least horizontal force, applied at the axle, necessary to drag a wheel 100 cm. in diameter and weighing 25 kg. over an obstacle 10 cm. high. The weight of a wheel acts at its center. [18.75 kg. wt.]

12.—A window sash 3 ft. wide and weighing 25 lb. is supported by two sash cords to each of which is attached a piece of iron weighing 10 lb. If one of the cords is broken, find at what distance from the middle of the sash the hand must be placed to raise it with the least effort. [1 ft.]

13.—A uniform rod, 14 in. long and weighing 10 lb., is joined so as to be in the same straight line with another uniform rod 16 in. long and weighing 8 lb. Find the point on which they will balance. [0.33 in. from juncture.]

14.—A uniform beam 50 ft. long and weighing 100 lb. rests horizontally with its ends on two supports. The beam carries loads of 30, 50, and 80 lb. wt. at distances of 10, 20, and 35 ft. respectively from one support. Find the reaction at each support. [128 lb. wt.; 132 lb.wt.]

15.—Two men, one stronger than the other have to remove a block of stone weighing 270 lb. by means of a light plank 6 ft. long. Show where the block must be supported in order that the stronger man will bear 180 lb. wt. [2 ft. from stronger man.]

16.—The horizontal roadway of a bridge is 30 ft. long, weighs 6 tons, and rests on similar supports at its ends. What is the reaction at each support when a carriage weighing 2 tons is two thirds the way across? [4.33 tons wt.; 3.66 tons wt.]

17.—A 20 ft. pole with one end hinged at the ground is held in a vertical position by two ropes, one attached to the upper end and another at a point 4 ft. above the ground. The upper rope makes an angle of 30° with the horizon and is under a tension of 30 lb. wt. Find the tension in the other rope when horizontal. [129.9 lb. wt.]

§ 2. *Composition of Concurrent Forces.*

28. Definitions.—Two forces or systems of forces having identical effects on the motion of a body with respect to both translation and rotation are said to be *equivalent.* A single force equivalent to two or more forces is called the *resultant* of the set. A single force which, if added to a set of forces, will produce equilibrium, is termed the *equilibrant* of the system of forces. The resultant is equal in magnitude and opposite in direction to the equilibrant. The operation of finding the resultant of a set of forces is termed *composition of forces.* Forces are said to be *concurrent* when their lines of action pass through the same point.

29. *The Parallelogram of Forces.—Consider the two coplanar, non-parallel forces F_1 and F_2 acting on a rigid body at the points X and Y. It is required to find the direction and magnitude of their resultant. Prolong the lines of action of the two forces until they meet at some point A. Since the body is rigid, each force will produce the same effect if applied at any point in its line of action (Art. 13). Consequently the forces may be considered to be applied at the single point A.

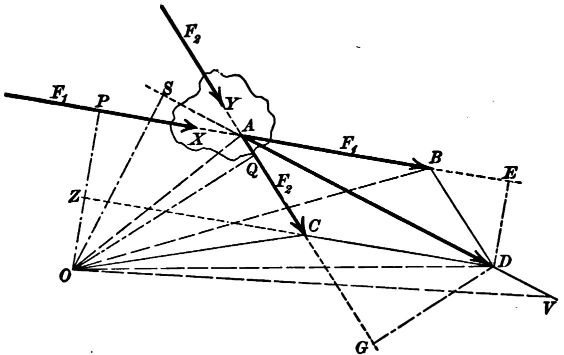

Fig. 15.

From the point A lay off the lines AB and AC representing, in direction and in length, the direction and magnitude of the forces F_1 and F_2. From the theorem of moments (Art. 25) it follows that the sum of the moments of the three forces F_1, F_2 and their equilibrant, about any axis normal to their plane must equal zero. Whence, the sum of the moments of F_1 and F_2 about any axis normal to their plane equals the moment of their resultant about the same axis. For an axis passing through the point A, the moments of both F_1 and F_2 are zero: consequently for this axis the moment of their resultant is zero. Therefore, either the resultant passes through the point A, or its magnitude is zero. If, however, F_1 and F_2 are not equal and oppositely directed, their resultant is different from zero. Whence, when F_1 and F_2 are not equal and oppositely directed, they have a resultant and this resultant passes through A.

By locating a second point through which the resultant passes, the direction of the resultant will be determined. Draw CD and BD parallel respectively to F_1 and F_2. From their point of intersection, D, draw DE and DG normal respectively to F_1 and F_2. Taking moments about D, we have

$$-F_1(DE) + F_2(DG) = -\text{area } (ABDC) + \text{area } (ABDC) = 0.$$

Whence, the moment of the resultant of F_1 and F_2 about the point D equals zero. Therefore the resultant passes through the point D. It has been already shown to pass through the point A. Consequently, if two concurrent forces be represented in direction and magnitude by two right lines, the resultant of the two forces is represented in *direction* by the diagonal of the parallelogram formed with these two lines as sides(¹).

The *magnitude* of the resultant is now to be determined. Let the line AV represent the resultant whose magnitude is required. It has been shown above that the magnitude of AV must be such that the moment of AV about any axis equals the sum of the moments of F_1 and F_2 about the same axis.

From any point, O, in the plane of F_1 and F_2 draw lines OP, OQ, and OS, normal respectively to F_1, F_2, and R', where R' represents the unknown resultant. Also draw the lines OA, OB, OC, OD, and OV.

---

(1)That the resultant R necessarily lies in the plane of F_1 and F_2 may be shown as follows:—If R did not lie in the plane of F_1 and F_2 it would either be parallel to that plane or its direction would cut that plane in some point A. In the first case choose any line in the plane of F_1 and F_2, in the second case any line in that plane except lines which pass through A. Then the moments of F_1 and F_2 about this chosen line will each be zero, whereas that of R will be finite. That is, the moment of R about the chosen line is different from the sum of the moments of F_1 and F_2 about that line.

The condition of equal moments about an axis normal to the plane of F_1 and F_2 passing through any point O can be written

$$R'(OS) = F_1(OP) + F_2(OQ), \qquad (8)$$

or, expressing the forces in terms of the lines in the diagram representing them,

$$AV(OS) = AB(OP) + AC(OQ), \qquad (9)$$

But
$$\begin{aligned} AV(OS) &= 2 \text{ area of triangle } (OAV), \\ AB(OP) &= 2 \text{ area of triangle } (OAB), \\ AC(OQ) &= 2 \text{ area of triangle } (OAC). \end{aligned}$$

Whence (9) can be put into the form

$$\text{area } (OAV) = \text{area } (OAB) + \text{area } (OAC). \qquad (10)$$

But $\text{area } (OAB) = \tfrac{1}{2} [(AB)(PO)] = \tfrac{1}{2} [AB(PZ + ZO)]$ or, since by construction $CD = AB$,

$$\begin{aligned} \text{area } (OAB) &= \tfrac{1}{2} (CD)(PZ) + \tfrac{1}{2} (CD)(ZO) \\ &= \text{area } (ACD) + \text{area } (OCD). \end{aligned}$$

On substituting this value in (10), the condition of equal moments assumes the form,

$$\text{area } (OAV) = \text{area } (ACD) + \text{area } (OCD) + \text{area } (OAC). \qquad (11)$$

But this equation is true only when

$$\text{area } (OAV) = \text{area } (OAD),$$

that is, when $AV = AD$. Consequently, if two co-planar, non-parallel forces be represented in direction and magnitude by two right lines, the resultant of the two forces is represented in *magnitude* by the diagonal of the parallelogram formed with these two lines as sides.

It has now been shown that, *if two adjacent sides of a parallelogram represent in magnitude and direction two concurrent forces, both acting either toward or away from their point of intersection, then the diagonal of the parallelogram passing through their intersection represents the resultant of the forces both in magnitude and in direction.* This important proposition is called the Law of the Parallelogram of Forces.

30. Computation of the Resultant of two Concurrent Forces.—Let AB and AC represent in direction and in magnitude two forces F_1 and F_2 whose lines of action are inclined to one another at an angle Φ. Then the resultant of the forces will be represented by the diagonal AD of the parallelogram erected on the lines AB and AC. The magnitude and direction of the resultant, R, will now be determined.

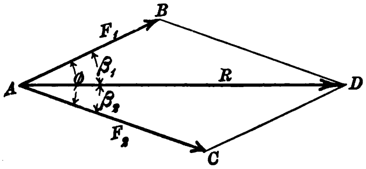

Fig. 16.

From elementary trigonometry,

$$R^2 = F_1^2 + F_2^2 - 2\,F_1\,F_2 \cos ABD,$$

or, $$R^2 = F_1^2 + F_2^2 + 2\,F_1\,F_2 \cos \Phi. \tag{12}$$

This equation gives the magnitude of the resultant of two concurrent forces in terms of the magnitude of the two given forces and the angle between them.

31. When $\Phi = 0$, then $\cos \Phi = 1$ and (12) becomes

$$R = F_1 + F_2, \tag{13}$$

that is, the resultant of two parallel forces having the same direction lies in their plane, is parallel to them, and in magnitude equals their sum.

When $\Phi = 180°$, then $\cos \Phi = -1$, and (12) becomes

$$R = F_1 - F_2, \tag{14}$$

that is, the resultant of two oppositely directed parallel forces lies in their plane, is parallel to them, and in magnitude equals the difference between them.

Consequently, *the resultant of any two parallel forces lies in their plane, is parallel to them, and in magnitude equals their algebraic sum.*

32. The direction of the resultant of two forces can be found as follows: From Fig. 16,

$$\frac{\sin \beta_1}{F_2} = \frac{\sin \beta_2}{F_1} = \frac{\sin \Phi}{R}.$$

Therefore

$$\sin \beta_1 = \frac{F_2 \sin \Phi}{R} \tag{15}$$

and

$$\sin \beta_2 = \frac{F_1 \sin \Phi}{R}. \tag{16}$$

33. Resultant of Any Number of Concurrent Forces.—Let F_1, F_2, F_3, etc., be any number of concurrent forces. Let the resultant of F_1 and F_2 be R_1. R_1 can now be compounded with F_3 giving a resultant R_2. Manifestly R_2 is the resultant of F_1, F_2, and F_3. Proceeding in like manner with the other forces, the resultant of the whole system can be obtained. The following two paragraphs show how this method can be considerably abbreviated.

34. The Triangle of Forces.—From the law of the parallelogram of forces, the resultant of two forces F_1 and F_2 applied at any point A, (Fig. 17), is given by the diagonal AD of the parallelogram constructed on these two sides. In the figure, since the line BD is in the same direction and has the same length as the line AC, the forces F_1, F_2, and R are represented in magnitude and direction by the sides of the triangle ABD.

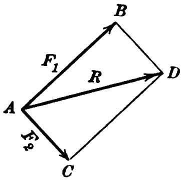

Fig. 17.

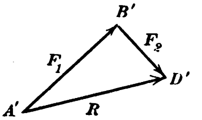

Fig. 18.

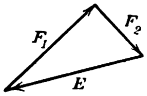

Fig. 19.

Whence, if the lines A' B' and B' D' (Fig. 18) are drawn in the directions of the concurrent forces F_1 and F_2 and of lengths proportional to their respective magnitudes, then the line $A'D'$ closing the triangle will represent in magnitude and direction the resultant of the two forces.

Since the equilibrant of a system of forces is equal in magnitude and is in the opposite direction to the resultant, the forces F_1, F_2, and their equilibrant E, would be represented as in Fig. 19. Consequently, *if three concurrent forces are represented in magnitude and direction by the three sides of a triangle described in succession, the forces are in equilibrium.* This proposition is called the Triangle of Forces.

35. The Polygon of Forces.—By an extension of the method used in the preceding paragraph it can be shown that if any number of concurrent forces are represented in direction and in magnitude by lines forming sides of a continuously described polygon, the line drawn from the starting point to close the polygon represents the resultant in magnitude and direction. Consequently, *if a number of concurrent forces be represented in direction and magnitude by the sides of a closed polygon taken in order, the forces are in equilibrium.* This proposition is called the Polygon of Forces.

PROBLEMS.

1.—A piece of wire 26 in. long and strong enough to support directly a load of 100 lb. wt., is attached to two points 24 in. apart in the same horizontal line. Find the maximum load that can be suspended at the middle of the wire. [76.92 lb. wt.]

2.—A string 7 ft. long has its ends attached to two points in the ceiling 5 ft. apart. When a stone is attached to the string 3 ft. from one end there is in the short portion of the string a tension of 8 lb. wt. and in the longer portion a tension of 6 lb. wt. Find the weight of the stone. [10 lb. wt.]

3.—Three cords of equal length are attached to a small iron ring. Two boys, at a distance from one another equal to the length of each cord, pull at the ends of two of the cords with forces of 50 and 60 lb. wt. respectively. What force must a third boy exert on the other cord so that the ring will be at rest? Find also the angle the third cord makes with the cord in which there is the tension of 60 lb. wt. [95.3 lb. wt.; about 27°.]

4.—A boat is towed along the middle of a canal 50 ft. wide by mules on both banks. Each rope is 72 ft. long and is under a tension of 800 lb. wt. Find the total effective pull on the boat. [1134.8 lb. wt.]

5.—Three smooth posts fixed in the ground so as to form an equilateral triangle are wrapped about by a tightly stretched rubber band which is under a tension of 2 lb. wt. Find the force acting on each post. [2 $\sqrt{3}$ lb. wt. toward the center of the triangle.]

6.—Two boys strike a foot ball at the same instant, one with a force of 25 lb. wt. acting north, and the other with a force of 30 lb. wt. acting east. What direction will the ball take? [About 39° 48′ north of east.]

7.—When the cord sustaining a pictur [illegible] is hung over a nail the two halves of the cord make an [illegible] one [illegible] sustain a tension of 10 lb. wt. W[illegible] frame? [17.3 lb. wt.]

§ 3. *Resolution of Forces.*

36. Definitions.—Two forces which, acting together, are equivalent to a single force are called the *components* of the given force. The operation of finding two components of a given force is called *resolution* of the force.

Since a straight line can be the diagonal of an indefinite number of parallelograms, it follows that any force can be the resultant of an indefinite number of pairs of components. For instance, in Fig. 20, R is the resultant of F_1 and F_2; in Fig. 21, R is the resultant of two different components F_3 and F_4; while in Fig. 22, R is resolved into still another pair of components, F_5 and F_6.

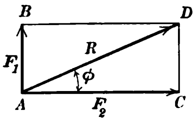

Fig. 20.

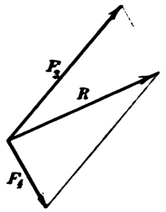

Fig. 21.

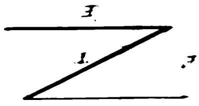

Fig. 22.

If, however, the directions of the two components be assigned, then there is but one pair of forces equivalent to the given force. The most useful resolution is into components at right angles to one another. In this case the required components are called the *rectangular components* of the given force.

37. Rectangular Components of a Force.—To fix the ideas let it be required to find the vertical and horizontal components of a force represented in direction and magnitude by the line AD, (Fig. 20). With this line as a diagonal construct a parallelogram having its opposite sides vertical and horizontal respectively. From the law of the parallelogram of forces, it follows that if the lines AB and AC represent in direction and in magnitude two forces F_1 and F_2, then the diagonal AD represents the direction and magnitude of their resultant. Conversely, AB and AC represent the two components of R in the required directions.

Let the angle between R and F_2 be called Φ. Then the vertical component of R is

$$F_1 = R \sin \Phi \tag{17}$$

and the horizontal component of R is

$$F_2 = R \cos \Phi. \tag{18}$$

Consequently, the rectangular component of a force represented by the line AD, in any given direction, AC, is represented in magnitude and direction by the orthogonal projection of AD upon AC.

As forces are seldom resolved in any directions except those perpendicular to one another, the terms "component of the force" and "resolved part of the force" are commonly used to denote the *rectangular* component of the force in the assigned direction.

38. The Component, in any direction, of the Resultant of a System of Concurrent Forces.—Let AB, AC, and AD, represent the forces F_1, F_2, and their resultant, respectively. It is required to find the relation between the resolved parts of the two forces in some assigned direction XX', and the resolved part of their resultant in the same direction.

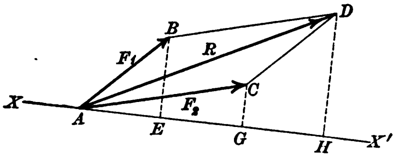

Fig. 23.

From, B, C, and D draw perpendiculars to XX'. The components of F_1, F_2, and R in the direction XX' are represented by the lines AE, AG, and AH, respectively.

Since AE and GH are orthogonal projections on the same straight line of two equal parallel lines, they are equal. Therefore,

$$AH\ [= AG + GH] = AG + AE.$$

Consequently, the sum of the components in any assigned direction of two concurrent forces, equals the component in that direction of the resultant of the two forces.

By a simple extension of the method used above, it can be *shown that the sum of the components in any assigned direction of any system of concurrent forces, equals the component in that direction of the resultant of the system of forces.*

This theorem furnishes an additional method of determining the resultant of a set of concurrent forces.

39. Illustration of the Resolution of Forces in the sailing of a boat. —Consider the case of a sailboat sailing into the wind. Let *CA* be the force of the wind on the sail. One part of this force, that is, the component *BA* parallel to the sail, is ineffective in moving the boat. The component *CB* of the force *CA*, acting perpendicular to the sail, produces a pressure on the sail. Resolve *CB* into components parallel and normal to the keel of the boat. The component *DB* is the only part of the force of the wind acting in the direction of the keel of the boat; that is, it is the only part of the force of the wind effective in propelling the boat forward. The component *CD* tends to displace the boat sideways. This tendency is partly prevented by the pressure of the water against the keel and side of the boat. This pressure acts at some point *O*, and the boat is so designed that the center of pressure of the wind is slightly aft from *O*. The wind and water together, then, tend to turn the boat so that it heads more nearly into the wind. This tendency is overcome by the rudder.

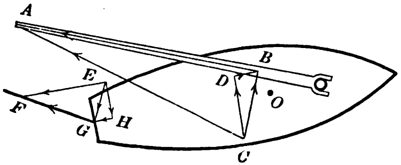

Fig. 24.

The force acting on the rudder will be represented by *EF*. Of this force, the component *GF* parallel to the rudder exerts no action on the rudder. The normal component *EG* produces a pressure on the rudder. This pressure *EG* may be resolved into two components *HG* and *EH*, parallel and normal respectively to the keel of the boat. The component *HG* acts as a resistance to the boat's motion. The component *EH* tends to push around the stern of the boat. When *CD* is the equilibrant of *EH* and the pressure of the water at *O*, the boat is in equilibrium and moves with uniform speed.

PROBLEMS.

A road is said to have an x % grade when the tangent of its inclination to the horizontal equals x.

1.—A horse is attached to a wagon so that the traces make an angle of 20° with the ground. If the road is level and offers a resistance to the wagon's motion of 30 lb. wt. find how much a horse must pull in order to keep the wagon moving uniformly. [31.92 lb. wt.]

2.—A man who can exert a force of but 50 lb. wt. is required to load a 200 lb. barrel into a wagon whose bed is 3.5 ft. high. Find how long a plank he must use in order that he can roll the barrel into the wagon. [14 ft.]

3.—A balloon capable of supporting a body weighing 200 kg. is held by a rope which makes an angle of 60° with the horizontal. Find the tension of the rope and the horizontal pressure of the wind on the balloon. [231 kg. wt.; 115.5 kg. wt.]

4.—A picture weighing 25 lb. is suspended from a nail in the wall by means of a wire whose ends are fastened to the sides of the frame. Find the tension in the wire : first, when the two halves of the wire make an angle of 60°; second, when this angle is 30°. [14.43 lb. wt.; 12.9 lb. wt.]

5.—A body weighing 125 kg. is suspended by a rope. A second rope attached to the body is drawn in a horizontal direction until the suspending rope has been deflected 30° from the vertical. Find the tensions in the two ropes. [144.3 kg. wt.; 72.15 kg. wt.]

6.—A horse exerts a force of 50 lb. wt. in towing a boat along a canal. If the boat is 10 ft. from the towpath and moves parallel to it, find the effective force of the horse when the towline is 20 ft. long. [43.3 lb. wt.]

7.—In a certain steam engine the piston pressure is 22,500 lb. wt. The connecting rod makes a maximum angle of 15° with the line of action of the piston. Find the maximum pressure on the guides. [6027.75 lb. wt.]

8.—If 150 lb. wt. per ton is a sufficient tractive force to draw a loaded wagon along a level road, what tractive force per ton will be required to draw the load up a 10 per cent. grade on the same road? [350 lb. wt.]

9.—A fish caught by a rod and line pulls with a force of 4 lb. wt. The inclination between the rod and line is 50°. What force on the end of the rod normal to its length must the rod be able to bear? [3.064 lb. wt.]

10.—A board will just support a load of 200 lb. wt. placed at its middle point when the board is inclined to the horizon at an angle of 25°. What weight would it support when placed horizontally? [181.26 lb. wt.]

§ 4. *Composition of Parallel Forces.*

40. Resultant of Two Parallel Forces.—It has been shown, (Art. 31), that the resultant of two parallel forces lies in their plane, is parallel to them, and in magnitude equals their algebraic sum. From the theorem of moments, (Art. 25), it follows that the line of action of the resultant must be such that its moment, about any axis perpendicular to the plane of the forces, is equal to the sum of the moments of the two forces about the same axis.

Two cases should be distinguished: first, when the two forces are in the same direction; and, second, when they are in opposite directions.

(*a*). *When the two Forces are in the Same Direction.* Let two forces F_1 and F_2 be applied at two points A and B of a rigid body. Let R_1 be the resultant of these two forces. It is required to find P, the point of application of this resultant.

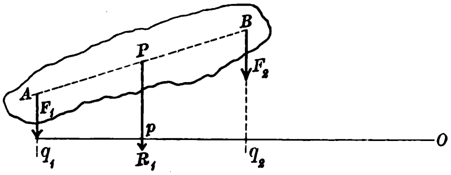

Fig. 25.

Consider moments about an axis, normal to the plane of the two forces, passing through any given point O. Draw a line from O intersecting F_1, F_2, and R_1 normally at the points q_1, q_2, and p. Then,

$$F_1\,(q_1\,O) + F_2\,(q_2\,O) = R_1\,(p\,O) = (F_1 + F_2)\,(p\,O),$$

$$F_1\,(q_1\,O - p\,O) = F_2\,(p\,O - q_2\,O),$$

$$F_1\,(q_1\,p) = F_2\,(p\,q_2),$$

$$F_1\,(AP) = F_2\,(PB). \tag{19}$$

(*b*). *When the two Forces are in Opposite Directions.* Consider moments about an axis, normal to the plane of the two forces, passing through any given point O. Then

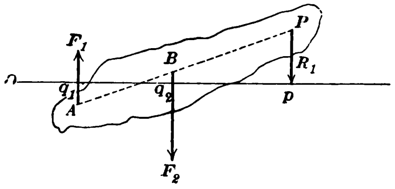

Fig. 26.

$$F_1(q_1 O) - F_2 (q_2 O) = - R_1(p O) = - (F_2 - F_1) (p O),$$

$$F_1 (q_1 O - p O) = F_2 (q_2 O - p O).$$

Multiplying throughout by $(- 1)$ we obtain

$$F_1 (p q_1) = F_2 (p q_2),$$

$$F_1 (A P) = F_2 (PR). \qquad (20)$$

41. *Resultant of a System of Parallel Forces.—The resultant of any system of parallel forces F_1, F_2, F_3, etc. having fixed points of application, A, B, C, etc., can be determined by an application of the theorem demonstrated in Art. 40. Thus, divide the distance AB into two segments such that $F_1(AP) = F_2(PB)$. Then P is the point of application of the resultant of F_1 and F_2. The magnitude of this resultant, R_1, equals the sum of the two forces.

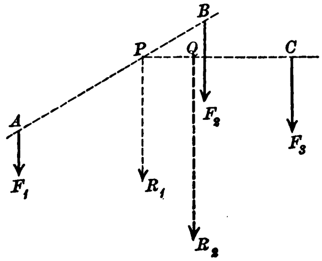

Fig. 27.

Again, the point of application of the resultant of R_1 and F_3 is found to be at some point Q on the line PC such that $R_1 (PQ) = F_3 (QC)$. Consequently, the magnitude of the resultant, R_2, of the three parallel forces F_1, F_2, and F_3 equals their sum, and its point of application is at Q.

Proceeding in this manner, the resultant of any number of parallel forces having fixed points of application can be determined.

42. The Force Couple defined.—No single force can replace two oppositely directed, parallel forces, having the same magnitude. If two such forces act in the same line, they will be in equilibrium. If, however, their lines of action are different, they will alter the angular motion of the body to which they are applied without affecting its linear motion.

A system composed of two equal, parallel, and oppositely directed forces having different lines of action is termed a *force couple*. A couple tends to produce a change in the angular motion of the body to which it is applied, about an axis perpendicular to the plane of the forces constituting the couple. It is easily shown that the moment of a couple equals the product of one of the forces and the normal distance between the lines of action of the two forces constituting the couple, and is independent of the distance of the axis of rotation from the forces constituting the couple.

A couple can be held in equilibrium only by the application of another couple having an equal moment of the opposite sign. An unbalanced *force* produces a change in the *linear* motion of the body acted upon; an unbalanced *couple* produces a change in the *angular* motion of the body acted upon. The turning of a screw with a screw driver, and the turning of a bolt nut with a wrench, are examples of the action of force couples.

§ 5. *Equilibrium.*

43. Conditions of Equilibrium.—A body is in equilibrium when its motion does not change. In order that its linear motion may not change, the resultant force acting on the body must equal zero. In order that its angular motion may not change, the sum of the moments, about any axis, of all the forces acting on the body, must equal zero. If all the forces are in one plane, the first requirement is satisfied if the sum of the components of all the forces is zero for each of any two directions of resolution. Therefore, *if a body is in equilibrium under the action of a system of coplanar forces, the sum of the components of the forces is zero for each of any two directions of resolution, and the sum of the moments of the forces about any axis normal to their plane equals zero.*

The conditions of equilibrium of a system of coplanar forces can be expressed in the form of three equations. Two of these equations are obtained by resolving all the impressed forces in any two convenient directions, and equating separately the sum of the components in each direction to zero. The third equation is obtained by taking moments about an axis normal to their plane passing through any convenient point, and equating the sum to zero.

It is usually most convenient to make the resolutions in directions perpendicular to one another. In order to avoid, as much as possible, unknown forces appearing in the equation, it is advisable to make one of the resolutions normal to the unknown force, and to take moments about an axis passing through a point in its line of action. The value of these suggestions will appear in the solution of the following problem.

Problem.—A derrick has a uniform boom 20 feet long, weighing 150 pounds, with one end hinged to a vertical mast 20 feet from the top. A rope extends from the other end of the boom to the top of the mast. When the boom makes an angle of 60° with the mast and supports at its end a load of 1200 pounds' weight, find the tension in the rope, and the vertical and horizontal thrusts on the pin of the hinge connecting the boom to the mast.

Solution.—Since the boom is uniform, its weight may be considered to be all applied at its middle point. The boom is acted upon by five forces as shown in the diagram. Being in equilibrium, the sum of the moments of all the forces, with reference to any axis normal to the plane of the diagram, equals zero. By selecting for the axis a line normal to the plane of the forces and passing through the point O, the two unknown forces F_h and F_v will have zero moments, and so will not appear in the equation. Thus, taking moments about O,

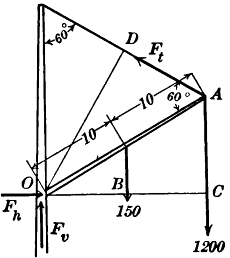

Fig. 28.

$$F_t\ (OD) - 150\ (OB) - 1200\ (OC) = 0,$$

or

$$F_t\ (20 \sin 60°) - 150\ (10 \sin 60°) - 1200\ (20 \sin 60°) = 0.$$

Whence $F_t = 1275$ lbs. wt.

Again, since the boom is in equilibrium, the sum of the components, in any direction, of all the forces acting upon it, equals zero. By resolving vertically, the unknown force F_h will not appear in the equation. Thus,

$$F_v + F_t \cos 60° - 150 - 1200 = 0.$$

Whence $F_v = 712$ lbs. wt.

Similarly, by resolving horizontally, neither the unknown force F_v nor the other vertical forces acting on the boom will appear in the equation. Thus,

$$F_h - F_t \sin 60° = 0$$

Whence $F_h = 1104$ lbs. wt.

44. Stability of Equilibrium.—If a body is at rest with respect to any selected point of reference, it is said to be in *static equilibrium.* A brick in a chimney is in static equilibrium with reference to the earth. A lump of mortar adhering to a falling brick is in static equilibrium with reference to every point of the brick, but is not in equilibrium with reference to any point of the earth. A body moving with either constant linear, or constant angular motion, is said to be in *kinetic equilibrium.*

Depending upon the behavior of a body in equilibrium after it has suffered a slight displacement, three classes of the state of equilibrium are distinguished. If after suffering a slight displacement the body recovers its former position, the equilibrium of the body is said to be *stable.* If, however, after receiving a slight displacement the body departs farther and farther from its former position, the equilibrium is termed *unstable.* If, after suffering a slight displacement, there is no tendency of the body either to recover its former condition or to depart farther from it, the body is said to be in *neutral* or *indifferent* equilibrium. The *degree of stability* of a body is measured by the amount of work necessary to effect a permanent change in the position or condition of the body.

A right cone resting with its base on a horizontal table is in stable static equilibrium; so, also, are a suspended plumb-bob and a body suspended by a vertical spiral spring. An egg balanced on an end, or a lead pencil on its point, are examples of unstable static equilibrium. A right cone lying on its side on a smooth horizontal table, or a wooden sphere floating in a basin of water, or a wheel on a smooth axle, are examples of neutral static equilibrium.

A boat being poled up a stream will be in stable equilibrium while the man is in the bow, but will be in unstable equilibrium if he poles from the stern. In order that a flying sky rocket shall not wobble, the point of application of the force acting upon it,—that is, the orifice from which the gases escape,—must be far forward. This is accomplished by having a long stick attached to the body of the rocket and extending backward from the fuse hole.

PROBLEMS.

1.—A uniform beam, 12 feet long and weighing 50 lb. rests with one end at the bottom of a vertical wall, while a point in the beam 10 ft. from the bottom is connected by a horizontal string with a point in the wall 8 ft. above the ground. Find the tension of the string and the pressure against the wall. [22.5 lb. wt.; 22.5 lb. wt.]

2.—A chandelier weighing 75 kg. is suspended at the intersection of two rafters inclined to one another at an angle of 120°. Find the thrust along each rafter due to the chandelier. [75 kg. wt.]

3.—Two uniform beams, each 24 ft. long and weighing 110 lb. joined at one end, rest with their other lower ends fixed to the tops of two vertical walls of the same height and 36 ft. apart. Find the horizontal thrust tending to overturn each wall. [62 lb. wt.]

4.—A uniform rod 3 ft. long and weighing 25 lb. is supported horizontally with one end hinged to a vertical wall and the other end attached by a string to a point 4 ft. above the hinge. A body weighing 50 lb. is suspended from the free end of the rod. Find the tension in the string and the horizontal thrust on the hinge. [78.1 lb. wt.; 46.87 lb. wt.]

5.—A uniform horizontal bar AB, 3 m. long and weighing 50 kg. has the end B hinged to the vertical side of a building, while the end A is supported by a light rope tied to it and to the building at a point 4 m. above B. Find the tension in the rope, and also the horizontal and vertical components of the reaction at the hinge. [31.25 kg. wt.; 18.7 kg. wt.; 25 kg. wt.]

6.—A body weighing 10 lb. is supported by two strings of lengths 3 ft. and 4 ft. respectively, attached to two points in the ceiling 5 ft. apart. Find the tension in each string. [8 lb. wt.; 6 lb. wt.]

CHAPTER III.

CENTROIDS.

45. The Centroid defined.—Let A and B, (Fig. 29), be the fixed points of application of the forces F_1 and F_2, and let P be the point of intersection of the line AB by the line of action of the resultant. So long as the two given forces remain parallel and continue to act in lines passing through A and B, then, however the body is displaced or however the direction of the forces may be changed, the line of action of the resultant will continue to pass through the same point P. Consequently, the point P is a definite point fixed in the body. This point has so many interesting properties that it has been given a name. The point of application of the resultant of a system of parallel forces is called the *centroid*, or the *center* of the system.

46 *Location of the Centroid of a System of Parallel Forces.—Although the method previously developed, (Art. 41), is of general application, the position of the centroid of a system of parallel forces having fixed points of application can usually be found more simply by a modification of this method.

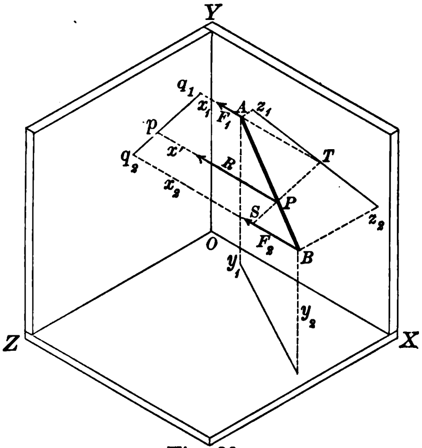

Fig. 29.

Consider a rigid body composed of a system of particles A, B, C, etc., acted upon by a system of parallel forces F_1, F_2, F_3, etc. Through any selected point of reference pass three planes that mutually intersect at right angles in the lines OX, OY, OZ. Let the distances of the points A, B, C, etc., from the plane YZ be represented by x_1, x_2, x_3, etc., while their distances from the XZ plane are y_1, y_2, y_3, etc., and their distances from the XY plane are z_1, z_2, z_3, etc.

Assume that the forces are first normal to the YZ plane then, in succession, normal to the other two coordinate planes. The magnitude of the resultant is the same in each case, and its line of action in each case passes through a common point fixed in the body called the centroid.

First consider two of the forces, F_1 and F_2, applied at the points A and B. It has been shown, (Art. 40), that the centroid, P, of the two forces F_1 and F_2 is so located that

$$F_1\ (AP) = F_2\ (BP).$$

Let q_1, q_2, and p be the intersections of the perpendiculars from A, B, and P with the plane YZ. Through P draw ST parallel to $q_2\, q_1$. Then, since the triangles ATP and PBS are similar,

$$AP : BP = AT : SB.$$

Consequently $$F_1\ (AT) = F_2\ (SB).$$

Let the distance of the centroid of F_1 and F_2 from the plane YZ be denoted by x'. Then, since

$$x' = (p\, P) = (q_1\, T) = (q_2\, S),$$

the above equation can be written in the form

$$F_1\ (x' - x_1) = F_2\ (x_2 - x'),$$

or, $$(F_1 + F_2)\ x' = F_1\, x_1 + F_2\, x_2.$$

The same process may be applied to the sum of $(F_1 + F_2)$ and F_3, and so on indefinitely. Thus if X represents the limit of the distance from the plane YZ of the centroid of the forces $F_1, F_2, F_3, \ldots . F_n$, (where the number of forces, n, is indefinitely large), we obtain,

$$(F_1+F_2+F_3+ \ldots F_n)\ X = F_1\, x_1+F_2\, x_2+F_3\, x_3+ \ldots F_n\, x_n$$

or $$X = \lim_{n=\infty} \frac{\Sigma\, Fx}{\Sigma\, F}. \tag{21}$$

In the same manner for the two other coordinate planes,

$$Y = \lim_{n=\infty} \frac{\Sigma\, F y}{\Sigma\, F}, \tag{22}$$

$$Z = \lim_{n=\infty} \frac{\Sigma\, F z}{\Sigma\, F}. \tag{23}$$

47. The Center of Gravity defined.—"If the action of terrestrial or other gravity on a rigid body is reducible to a single force in a line passing always through one point fixed relatively to the body, whatever be its position relative to the earth or other attracting mass, that point is called its *center of gravity.*"(1).The only bodies which have true centers of gravity are uniform spherical shells, uniform spheres, and spheres whose density either increases or diminishes uniformly from the center to the circumference. In other cases the line of action of the weight of the body does not pass through the same point when the position of the body is changed.

Although when a body is acted upon by a system of non-parallel forces there will be no invariable point of application of the resultant as the position of the body is changed, the departure from parallelism of the gravitational forces acting on the different parts of a body is so small that it is customary, especially in engineering problems, to assume that there is a definite point fixed in the body at which the weight of the body is applied. Thus, it is assumed that every body has a center of gravity coincident with that point at which the resultant of the gravitational forces acting on the body would be applied if they were parallel. Custom having sanctioned this loose use of the term, it will occasionally be employed in the succeeding pages.

To emphasize the fact that the point to which the weight of an unsymmetrical body is applied is different for different positions of the body with respect to the earth, it will be well to consider the case of a body of very great length and small width. The calculation of the problem will be much simplified if the body is imagined to consist of two spheres connected by a rod whose weight can be neglected.

Let the two spheres, A and B, when on the surface of the earth, have the weight of 100 and 1000 pounds, respectively. Let the distance between their centers be equal to the radius of the earth, 4000 miles. With the smaller sphere resting on the surface of the earth, find the position of the point of application of the weight of the body in two cases—(1) when the line connecting the centers of the two spheres, AB, makes with AO, the radius of the earth passing through the smaller sphere, an angle of 90°; (2) when the line connecting the centers of the two spheres makes with the same radius an angle of 140°.

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(1) Thompson & Tait, Natural Philosophy, II p. 78.

Newton discovered that the weight of a body, due to the attraction between it and the earth, is directed toward the center of the earth, and that its magnitude varies inversely as the square of the distance of the body from the center of the earth.

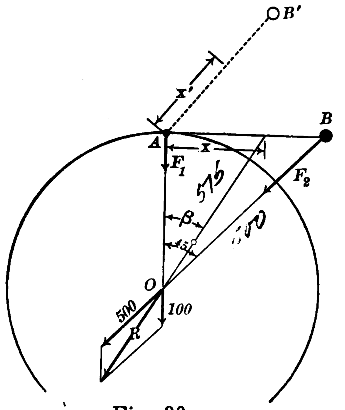

Fig. 30.

In the first case, then, since the distance BO is [4000 $\sqrt{2}$ =] 5656 mi., it follows that the weight of the sphere at B, directed toward O, has the magnitude

$$F_2 = 1000 \cdot \left(\frac{4000}{5656}\right)^2 = 500 \text{ lbs. wt.}$$

Imagine the weights of both spheres applied (see Art. 13) at O. Then by (12) the value of their resultant is given by

$$R^2 = 100^2 + (500)^2 + 2\ (100)\ 500 \cos 45°.$$

Whence, $R = 575$ lbs. wt.

By (15), $\sin \beta = \dfrac{500 \sin 45°}{575}$.

Whence, $\beta = 37.°9$.

From the figure, $\dfrac{x}{AO} = \tan \beta$.

From this, on substituting the values of AO and β, $x = 3120$ mi.

In the second case, the position of the body is that represented by AB'. A solution similar to that for the first case shows that $x' = 2400$ mi. If the change in weight as a body is moved away from the earth were to be neglected, the assumption being made that B, wherever it is, weighs 1000 lbs., the values obtained for x and x' respectively would be 3500 and 3370 mi.

Whether, then, the change in weight of a body is neglected or not, the above problem shows that the point of application of the weight changes as the body is moved.

48. *Location of the Point of Application of the Weight of a Body in a few simple cases.—In most cases the location of the point of application of the weight of a body requires more powerful mathematical methods than are at the command of the students for whom this book is intended. Some of the most useful cases, however, can be solved very simply. Assuming that the gravitational forces acting on all parts of a body act in parallel lines, we will now determine the position
body act in parallel lines, we will now determine the position
of the point of application of the weight of a few uniform bodies having simple geometrical form.

(*a*). *Uniform thin Rod.* Imagine the length l of an indefinitely thin rod to be divided into a very large number n of elements each of length Δl. Let the weight of each of these very small elements be denoted by F. Take as a plane of reference, from which distances are to be measured, one perpendicular to the rod at a distance b from one end. Then, from (21), the distance from this plane, of the centroid of the system of parallel forces acting on the rod, that is, the distance of the point of application of the weight of the rod, is

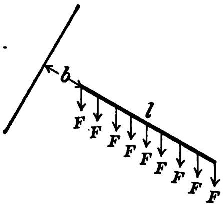

Fig. 31.

$$X = \lim_{n=\infty} \frac{\Sigma\, F x}{\Sigma\, F}$$

$$= \lim_{n=\infty} \frac{F b + F[b+\Delta l] + F\,[b + 2\Delta l] + \ldots + F[b+(n-1)\Delta l]}{n\,F}$$

$$= \lim_{n=\infty} \frac{1}{n}\ \ [b + [b+\Delta l] + [b + 2\Delta l] + \ \ldots\ + [b + (n-1)\Delta l]$$

$$= \lim_{n=\infty} \frac{1}{n}\cdot\frac{n}{2}\ \ [2\,b + (n-1)\Delta l] = \lim.\, [b + \tfrac{1}{2}\,(n\,\Delta\,l - \Delta\,l)]$$

But $n\Delta l$ equals the length of the rod. In the limit when n is taken indefinitely large, the size of each element becomes vanishingly small, i. e. $\Delta\, l = 0$. Consequently

$$X = b + \tfrac{1}{2}\, l, \qquad (24)$$

Therefore, the weight of an indefinitely thin uniform rod is applied at its middle point.

(*b*). *Uniform plane Lamina bounded by a parallelogram.* Imagine the parallelogram $ABCD$ to be divided into a

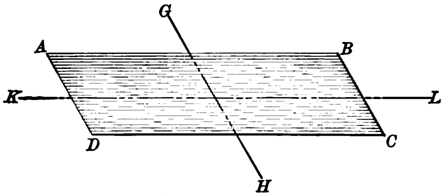

Fig. 32.

series of very many narrow strips parallel to the side AB. Since the weight of each strip is applied at its middle point, the point of application of the weight of the lamina must be somewhere on the line *GH* bisecting these strips. Similarly, by imagining that the parallelogram is divided into a series of very narrow strips parallel to the side *AD*, it can be shown that the point of application of the weight of the lamina must be somewhere on the line *KL* bisecting these strips.

Therefore, the weight of a uniform plane lamina bounded by a parallelogram is applied at the intersection of its diagonals.

(*c*). *Uniform plane Lamina bounded by a regular polygon*. A regular polygon is a polygon which is both equilateral and equiangular. The circle is a regular polygon having an infinite number of sides.

Proceeding exactly as in the preceding case, it can be shown that the weight of a uniform plane lamina bounded by any regular polygon is applied at its geometrical center, i. e. at the intersection of the bisectors of its angles.

(*d*). *Uniform Prisms and Cylinders.* A prism is a solid whose ends are similar equal and parallel plane figures and whose lateral faces are parallelograms. Any section of a prism parallel to a lateral edge is a parallelogram, while any section normal to a lateral edge is a regular polygon. A prism having a circle for its normal section is called a circular cylinder.

Consider the regular prism *AJ* to be divided into a large number of thin laminae by a series of planes parallel to one of the lateral faces, as *AH*. Since the point of application of the weight of each lamina is at its geometrical center, the weight of the entire prism is applied at some point on the line *MN* drawn through the centers of this series of laminae. The line *MN* is in a plane parallel to the ends of the prism.

Again, consider the prism to be divided into a series of thin laminae by planes parallel to some other face, as *BI*. The point of application of the weight of the prism is now seen to be at some point on the line *OQ* drawn through the centers of this series of laminae. This line is in a plane parallel to the ends of the prism.

Since the lines MN and OQ are in the same plane, they intersect at some point P. Since the point of application of the weight of the prism is on both the lines MN and OQ, it must be at their point of intersection. It is easily proven by geometry that $MP = NP$ and $QP = OP$. Consequently *the point of application of the weight of a prism (or cylinder) is at its geometrical center.*

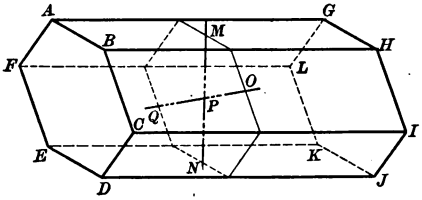

Fig. 33.

(*e*). *Uniform Sphere.* Consider the sphere to be divided into a large number of circular laminae by a series of parallel planes normal to any diameter. The point of application of the weight of each of the laminae being at its center, it follows that the point of application of the weight of the sphere is somewhere on the diameter normal to the laminae. By repeating this process for another diameter, it can be shown that *the point of application of the weight of the sphere is at the intersection of the two diameters, that is, at the geometrical center of the sphere.*

CHAPTER IV.

Friction between Solids.

§ 1 *Static, Kinetic, and Rolling Friction.*

49. Reaction at a Smooth Surface.—A surface which offers no resistance to the sliding of a body along it, is said to be *smooth.* Smooth surfaces are ideal but some surfaces are so nearly smooth that for practical purposes they may be considered to be quite so.

The reaction exerted by a smooth surface is always normal to the surface, for if it were not normal there would be a component of the reaction that would oppose motion in some direction—that is, the surface would not be smooth.

50. Coefficient of Static Friction.—Consider a body resting on a horizontal plane, (Fig. 34). The body is in equilibrium under the action of the force F_n normal to the plane (in this case, the weight of the body), and the reaction R of this force.

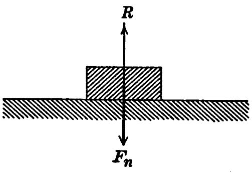

Fig. 34.

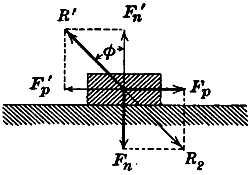

Fig. 35.

If another force, F_p, is applied parallel to the plane, the body may or may not move. If the surface is not smooth and the force F_p is not too great, the body will remain at rest. Under these conditions, the reaction R',(Fig. 35)must be equal and opposite to the resultant R_2 of the forces F_n and F_p. The component of the reaction normal to the surface equals the weight F_n of the body; while the component F_p' parallel to the surface equals the applied force F_p. The component of the reaction parallel to the surface is called the *static frictional resistance,* or the *static friction* between the two surfaces. The static friction can have any value between zero and a certain definite maximum value which is attained when the body is just on the verge of slipping. The force of friction has no independent existence of its own but is developed only through the action of an agent.

It is found by experiment that the force parallel to the surface of contact, required to cause the body to be just on the verge of slipping is directly proportional to the force normal to the two surfaces. Consequently

$$F_p = \mu F_n \tag{25}$$

where μ is the constant of proportionality depending upon the state and material of the surfaces in contact. The constant ratio of the force required to cause two bodies to be just on the point of slipping, to the force normal to the surfaces of contact, is called the *coefficient of static friction.* By experiment it is found that the coefficient of static friction depends upon the material and smoothness of the surfaces in contact, and is nearly independent of the thrust pressing them together and of their area of contact.

When F_p is greater than μF_n, the body slides; when F_p equals μF_n, the body is just in the point of sliding; when F_p is less than μF_n, the body is at rest.

The coefficient of static friction of leather on unpolished stone is so great that in walking on stone pavements there is little tendency to slip. The coefficient of friction between leather and ice, however, is so small that walking on ice is much less secure.

A *lubricant* is any substance which, coating with a thin film two bearing surfaces, will reduce their frictional resistance. On a dry day, the fine dust covering railroad rails acts as a lubricant to such an extent that there is a considerable tendency of the wheels of a locomotive to slip. After a rain has washed off the dust, the same locomotive can draw a heavier load.

Problem.—If, due to air resistance, friction of journals, etc., the motion of a train is opposed by a force of 5 pounds' weight per 1000 pounds, find the total weight of the train that can be hauled by a 100,000 pound locomotive on a level track such that the coefficient of static friction between the wheels and rails is 0.15.

Solution.—The greatest pull that can be exerted by the locomotive without its wheels slipping on the track is, from (25),

$$F_p = 0.15\ (100{,}000) = 15000 \text{ pounds' weight.}$$

If the weight of the locomotive and train be represented by F, then, from (25),

$$F_p = 0.005\ F.$$

Whence $F = 3{,}000{,}000$ pounds' weight.

Problem.—A uniform beam weighing 50 kilograms, inclined at an angle of 30° to the vertical, rests between a rough pavement and a smooth vertical wall and is just on the verge of slipping. Find the thrusts exerted against the pavement and the wall, and also find the coefficient of static friction between the beam and the pavement.

Solution.—Denote the length of the beam by l. Since the beam is uniform, its weight acts at its middle point. There are four forces acting on the beam as shown in the diagram. Since the beam is in equilibrium, the sum of the moments of all these forces, with reference to any axis normal to their plane, (Art. 43), equals zero. By selecting as the axis of moments, a line normal to the plane of the diagram and passing through the point A, the two unknown forces F_2 and μF_2 will have zero moments, and so will not appear in the equation of moments. Whence, taking moments about A,

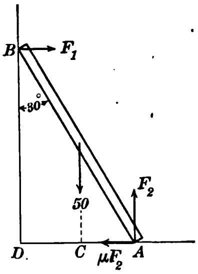

Fig. 36.

$$50\ (AC) - F_1\ (BD) = 0,$$

$$50\ (\tfrac{1}{2}\, l \sin 30^\circ) - F_1\ (l \cos 30^\circ) = 0.$$

Whence $F_1 = 14.4$ kg. wt.

Again, since the beam is in equilibrium, the sum of the components, in any direction, of all the forces acting upon it, equals zero. By resolving vertically, the unknown force of friction μF_2, as well as any other horizontal force, will not appear in the equation. Thus,

$$F_2 = 50 \text{ kg. wt.}$$

Similarly, by resolving horizontally,

$$F_1 - \mu F_2 = 0.$$

Whence

$$\mu = \frac{14.4}{50} = 0.29.$$

51. Limiting Angle of Friction.—When a body is just on the point of slipping, it is in equilibrium under the action of the three forces F_n, F_p, and R'. In this case the angle, Φ, between the reaction R' and the force F_n' normal to the surface of contact is given by the relation

$$\tan \Phi = \frac{F_p'}{F_n'} = \frac{F_p}{F_n} = \mu. \tag{26}$$

This value of Φ, namely, the greatest angle that can exist between the direction of the reaction and the perpendicular to the surface of contact, is called the *limiting angle of friction.* If a force be applied to a body in a direction making an angle with the perpendicular to the surface of contact less than the limiting angle of friction, no sliding can occur whatever may be the magnitude of the applied force.

For example, a steam engine pistonhead is attached to the piston-rod by means of a slender wedge called a key, k (Fig. 37). The angle of the key is made less than the critical angle of friction of iron on iron, so that no amount of force applied in the direction of the piston rod will dislodge the key from its position.

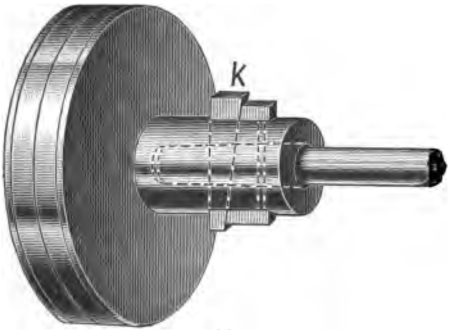

Fig. 37.

52. Coefficient of Kinetic Friction.—After a body has begun to slide, the motion is retarded by a frictional resistance which is somewhat less than it was just before slipping began. It is found that for ordinary pressures and speeds this *kinetic friction*, f_p, is proportional to the force, F_n, normal to the slipping surface. That is

$$f_p = b\,F_n, \tag{27}$$

where f_p equals the magnitude of the force in the direction of motion which is necessary to keep a body moving uniformly when acted upon by a force F_n normal to the slipping surface. The constant of proportionality, b, is called the *coefficient of kinetic friction.*

53. Illustrations distinguishing Static from Kinetic Friction.—When a train is in motion and the wheels are not slipping on the rails, there is static friction between the wheel and rail, and kinetic friction between the axle and bearings. When all the couplers of a long train of cars are tight, a locomotive may be unable to start a heavy train. But by first backing the locomotive until a number of couplers are loose, and then going ahead, the locomotive may be able to start in motion one car at a time until the whole train is moving. This is a case of overcoming piecemeal the static friction in the bearings of the train. When once in motion the large static friction is replaced by the smaller kinetic friction so that the train can be easily kept in motion.

Similarly, in stopping a train, the maximum effect of the brakes would be produced by tightening them on the wheels so that they would allow the wheels barely to turn within the brake shoes without the latter "seizing" the wheels, or the wheels slipping on the rails. The frictional resistance is greatest when slipping is just on the verge of occurring, without actually doing so.

54. Rolling Friction.—When a sphere or cylinder rolls on a plane surface there is developed, at the place of contact, a resistance to the motion which is called *rolling friction.*

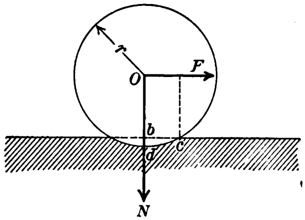

Fig. 38.

If the plane surface be of a yielding material that is permanently deformed by the force acting upon it, a wheel will sink into the surface. In this case, the resistance to motion is equivalent to the force required to cause the wheel constantly to mount an obstacle equal in height to the depression in the surface. Let N represent the normal force between the wheel and the surface; and let F be the force, applied at the axle parallel to the surface, required to keep the wheel rolling uniformly. Then, from the theorem of moments

$$F\ (Ob) = N\ (bc),$$

$$F = N\ \frac{(bc)}{(Ob)} = N\ \frac{(bc)}{r - (bd)},$$

or

$$F = \frac{N}{r}(bc) \text{ approx.} \tag{28}$$

Therefore, when a sphere or cylinder rolls on a yielding surface that takes a permanent deformation, the rolling friction is almost inversely proportional to the radius of the rolling body.

If the plane surface tends to resume its original shape as soon as the deforming force is removed, the problem is much more complicated than the one above considered.

The advantage of hauling a heavy load on a wagon instead of dragging the same load on the ground is partly due to the fact that rolling friction is less than sliding friction. For the same reason "friction bearings," using either rolling cylinders or balls, are extensively used in many classes of machinery.

PROBLEMS.

1.—If the coefficient of friction between the driving-wheels of a locomotive and the rail is 0.2, what must be the weight, in tons, of the locomotive in order to exert a pull of 5 tons' weight. [25 tons wt.]

2.—A uniform beam 36 ft. long weighing 150 lb. rests with one end against a smooth wall, and the lower end is prevented from slipping by a peg in the ground. If the inclination of the beam to the horizon be 30°, find the thrust against the wall and the vertical and horizontal components of the reaction at the peg. [129.9 lb. wt.; 150 lb. wt.; 129.9 lb. wt.]

3.—Determine the tractive force required to haul a car weighing 100 tons with constant velocity up a 2.5% grade when the coefficient of friction is 0.005. [3 tons wt.]

4.—A body weighing 50 lb. is held by friction on a plane inclined 30° to the horizontal. Find the frictional resistance and the pressure on the plane. [43.3 lb. wt.]

5.—A body weighing 60 lb. is just set in motion on a rough horizontal plane by a force of 9 lb. wt. parallel to the plane. If the force be withdrawn and the plane tilted, at what inclination of the plane to the horizon will the body begin to slide? [8°40′]

6.—A uniform horizontal beam 12 ft. long and weighing 60 lb. rests at a point 5 ft. from one end on top of a vertical post. A body weighing 80 lb. is hung from the end of the 7-ft. arm, and to the end of the 5-ft. arm is attached a rope 10-ft. long which has the other end fastened to the vertical post. Find the tension of the rope and the pressure on the post. If the beam be just on the point of slipping on the top of the post, find the coefficient of friction. [143 lb. wt.; 264 lb. wt.; 0.33.]

7.—A locomotive weighs 35 tons. The coefficient of friction between wheels and rails is 0.18. Find the greatest pull the engine can exert in pulling itself and a train. What is the total weight of itself and train which it can draw up a 1 per cent. grade if the resistance to motion on the level is 10 lb. wt. per ton? [6.29 tons wt.; 419.48 tons wt.]

8.—A uniform beam 20 ft. long and weighing 100 lb. rests with one end on a smooth floor, and the other end against a smooth vertical wall. If a string 5.4 ft. long connect the lower end with the foot of the wall, find the tension in the string, and the reactions exerted against the floor and the wall. [14 lb. wt.; 100 lb. wt.; 14 lb. wt.]

9.—A uniform beam rests against a smooth wall with its lower end on a smooth pavement. The lower end is prevented from slipping by a horizontal cord fastened to the wall. Find the reactions exerted against the wall and the pavement, and, also, the tension of the string. [½ mg cot θ; mg; ½ mg cot θ.]

10.—A uniform ladder weighs 100 lb., is 52 ft. long, and is inclined at an angle of 45° to a rough vertical wall and a rough horizontal floor. If the coefficient of friction at each end be 0.66, how far up the ladder can a man weighing 200 lb. ascend before the ladder will begin to slip? [46.5 ft.]

11.—A uniform beam weighing 60 kg. inclined at an angle of 60° to the horizon rests between a rough pavement and a smooth vertical wall. Find the reactions against the pavement and the wall, and also find the coefficient of friction between the beam and pavement when the beam is just on the point of slipping. [60 kg. wt.; 17.31 kg. wt.; 0.288].

12.—On top of a step ladder the sides of which make an angle of 60° with each other stands a man weighing 70 kg. Supposing each side of the step ladder is uniform and weighs 10 kg., find the reaction of the ground at the foot of each side of the ladder. If the feet of the ladder are just on the point of slipping, find the coefficient of friction between the ladder and the ground. [45 kg. wt.; 0.513.]

13.—A uniform ladder 70 ft. long and weighing 448 lb. is inclined equally to the ground and to a vertical wall. The coefficients of friction at the wall and at the ground are 0.33 and 0.5 respectively. A man weighing 160 lb. carrying a load of 64 lb. wt. ascends the ladder. How far can the man ascend before the ladder will slip? [49.9 ft.]

14.—A uniform beam resting with one end on a rough horizontal plane and the other end against an equally rough vertical wall is just on the point of slipping down when the angle between it and the horizontal is 30°. Find the coefficient of friction between the beam and the two equally rough surfaces. [0.577.]

15.—Two equal uniform boards, each 10 ft. long and weighing 20 lb., are hinged together at one end while the other ends rest on a smooth floor. If a string 5.4 ft. long connect the lower ends, find the tension in the string and the reaction at the hinges. [2.8 lb. wt.; 2.8 lb. wt.]

16.—A tent pole is kept in a vertical position by means of a guy rope fastened to the top and the other end looped around a peg driven into the ground. The tent pole is 6 ft. high and the peg is 3 ft. from the lower end. If the coefficient of friction between the rope and peg is 0.8, find the least inclination the peg can have to the vertical. [24° 46′.]

§ 2. *Mechanical Advantage.*

55. Definitions.—A *machine* is any mechanism which has for its purpose either the transformation of one form of energy into another form, or the change of a force in magnitude or direction. An electric motor transforms electrical energy into mechanical energy; a rope and pulley changes the direction of a force; a lever changes the magnitude of a force.

The ratio of the force developed by any machine, to the force applied, is called the *mechanical advantage* of the machine. Thus, in symbols,

$$\text{Mech. Adv.} = \frac{F_d}{F_a}.$$

The ratio of the work developed by any machine, to the work applied to it, is called the *efficiency* of the machine.

Or
$$\text{Eff.} = \frac{W_d}{W_a}.$$

56. The Wedge.—A rough isosceles wedge is in equilibrium under the action of a driving force on its base, and thrusts and frictional resistances on its slant faces. It is required to find the relation between the driving force on the base and the thrusts on the slant faces.

If the wedge is in equilibrium and is on the point of moving downward, the directions of the forces acting are as represented in Fig. 39. A force polygon can be constructed by drawing the lines, (Fig. 40), *ae*, *eg*, *gh*, *hj*, and *ja*, parallel respectively to the forces F, R', $\mu R'$, μR,

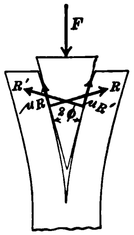

Fig. 39.

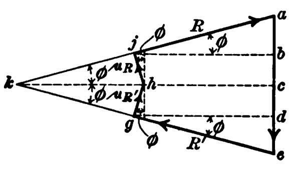

Fig. 40.

and R. If the angle of the wedge be denoted by 2Φ, $ake = 2\Phi$; and since μR and $\mu R'$ are normal respectively to R and R', μR and $\mu R'$ are inclined to the vertical at the angle Φ.

Resolving forces vertically,

$$\begin{aligned} F &= ab + bc + cd + de \\ &= R \sin \Phi + \mu R \cos \Phi + \mu R' \cos \Phi + R' \sin \Phi. \qquad (29) \end{aligned}$$

Resolving forces horizontally,

$$R \cos \Phi - \mu R \sin \Phi + \mu R' \sin \Phi - R' \cos \Phi = 0,$$

$$R (\cos \Phi - \mu \sin \Phi) = R' (\cos \Phi - \mu \sin \Phi)$$

$$R = R'. \qquad (30)$$

Equation (30) shows that in the case of an isoceles wedge having equal coefficients of friction on the two inclined faces, the thrusts exerted by the two inclined faces are equal.

Eliminating R' from (29) and (30),

$$F = R \sin \Phi + \mu R \cos \Phi + \mu R \cos \Phi + R \sin \Phi$$

$$F = 2 R \sin \Phi + 2 \mu R \cos \Phi.$$

Consequently

$$\frac{R}{F} = \frac{1}{2 (\sin \Phi + \mu \cos \Phi)}. \qquad (31)$$

This equation shows that the mechanical advantage of a wedge increases when either the angle of the wedge or the coefficient of friction is diminished.

57. The Screw.—If a portion of the thread of a screw be conceived to be unwrapped from the cylinder supporting it, the screw would have the appearance represented in Fig. 42. From this figure it is seen that the screw consists of an inclined plane wrapped about a cylinder. In fact, the screw is a combination of the lever with the inclined plane. The angle made by the thread with a plane perpendicular to the axis is called the angle of the screw. In the figure, the angle of the screw is represented by the symbol θ.

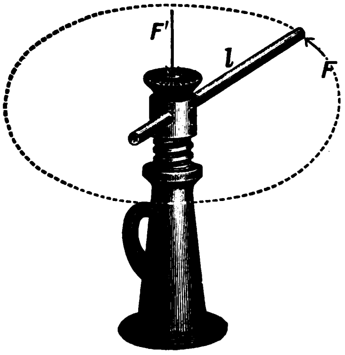

Fig. 41.

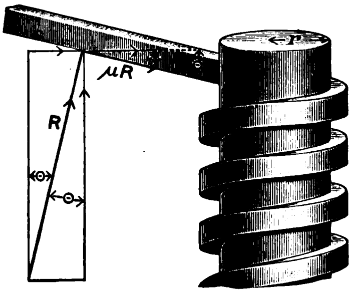

Fig. 42.

The mechanical advantage of the screw will now be determined. To fix the ideas, consider the ordinary jack-screw shown in Fig. 41. In this form the nut remains fixed while the screw advances axially as it is rotated in the nut. Let the radius of the screw be r, and let the force F applied horizontally at right angles to the lever arm of length l, be just able to overcome a force F' acting vertically downward on the screw. Denote the total normal reaction exerted by the nut on the screw by R, and the coefficient of friction between the screw and the nut by μ. The frictional resistance acting along the thread would then be μR.

When the vertical force F' is just on the point of being overcome, the screw is in equilibrium under the action of the force F' acting downward, the horizontal force F acting at a distance l from the axis of the screw, the normal reaction R acting at a distance r from the axis of the screw, and the frictional force μR acting along the thread.

Equating vertical forces, we have

$$F' = R \cos\theta - \mu R \sin\theta.$$

Equating moments of the horizontal forces about the axis of rotation, we have

$$Fl = (R \sin \theta)r + (\mu R \cos \theta)r.$$

Consequently the mechanical advantage of the screw is

$$\frac{F'}{F} = \frac{l (\cos\theta - \mu \sin\theta)}{r (\sin\theta + \mu \cos\theta)}. \qquad (32)$$

This equation shows that the mechanical advantage of a screw increases when either the pitch of the screw or the coefficient of friction decreases. If μ is small, the mechanical advantage is large even though $l = r$.

KINETICS, OR THE LAWS CONNECTING FORCE AND MOTION.

CHAPTER V.

The Motion of a Body under the Action of Zero Force.

§ 1. *Uniform Linear Motion.*

58. Linear Velocity and Speed defined.—By the motion of a body is meant the change of the position of the body *with reference to some other body*. Such expressions as *absolute rest* and *absolute motion* are meaningless. To describe a linear motion three elements must be specified. These are, the direction in which the displacement occurs, the distance traversed, and the time occupied in traveling this distance.

The rate of motion in any given direction is called *velocity*. Velocity has both magnitude and direction. Thus, we speak of a velocity of three miles per hour, north. This means that the body under consideration is moving toward the north at the rate of three miles each hour. When only the magnitude of a velocity is considered, it is called *speed*. Thus we speak of a horse being able to run a mile in two minutes. Here, the direction of the motion need not be considered. The speed of the horse is half a mile per minute.

A point which traverses equal spaces in equal times, however small the unit of time may be, is said to have uniform *speed*. When the speed is not uniform we often speak of the instantaneous speed at a given point. The *instantaneous speed* at any point is numerically equal to the distance the body would travel during the next second of time, if from that point the speed remained uniform. A point which preserves both its speed and the direction of its motion constant is said to be moving with *uniform velocity*. When either the direction of motion changes, or the distance passed over in equal times changes, the velocity is variable. For example, a body moving with constant speed in the circumference of a circle is moving with variable velocity.

59. The Unit of Linear Speed.—Denoting by s, the constant speed of a point traversing the distance x in time t, we have from the definition of speed,

$$s = \frac{x}{t}. \tag{33}$$

This equation shows that the unit of speed is such a speed that unit distance is traversed per unit of time. The unit of time employed in physics is the second. It follows that the unit of speed is either the *foot per second* or the *centimeter per second.* There are no generally accepted names for the units of speed.

60. The Composition of Uniform Linear Velocities.—A passenger moving across a moving railway carriage furnishes an example of what are called simultaneous velocities. The passenger has a velocity relative to a point of the carriage, and at the same time the carriage has a velocity with respect to a point on the earth. These are called *components* of the passenger's motion. The velocity of the passenger with reference to the earth is called his *resultant* velocity. If the component velocities of a body's motion are in the same direction, then the resultant velocity equals their algebraic sum.

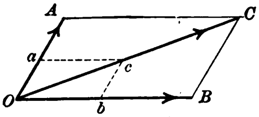

Fig. 43.

The resultant of two uniform velocities inclined to one another will now be determined. Let OB represent the velocity of the train relative to the earth and let OA represent the velocity of the man relative to the train. In other words, the line OB represents the direction and the distance traveled by the train, with reference to the earth, during some time t ; while OA represents the direction and the distance traveled by the man, with reference to the train, during the same time. During the time t, the man moved across the train from O to A. During this time, the line OA moved parallel to itself through the distance OB. Consequently at the end of the time t, the man has reached the point C.

It remains to be shown that throughout the time t, the man was moving along the diagonal OC. Suppose that during some interval of time t', less than t, the train has traversed the distance Ob; while, with reference to the train, the man has traversed the distance Oa. Draw ac and bc parallel to OB and OA, respectively. Following the method of the preceding paragraph, it is seen that at the end of the interval t' the man is at c.

But since the velocities along OA and OB are uniform,

$$t' : t = Oa : OA = Ob : OB,$$

or

$$bc : BC = Ob : OB.$$

Consequently Oc and OC are coincident. This means that at any instant during the time t, the man was on the diagonal OC. Therefore, *if two component linear velocities be represented by two adjacent sides of a parallelogram, the resultant velocity is represented by the diagonal which passes through their intersection.*

To illustrate the composition of two uniform linear velocities of magnitudes V_1 and V_2 in the directions AA' and BB', respectively, consider a stretched rubber band AB, (Fig. 44), having a paper wad fastened at some point W. If the end B remains fixed and the end A be moved in the direction AA', the paper wad will move in the parallel direction WC. If the point A move with such a speed that in unit time the point W traverses a distance X_1 proportional to the magnitude of the velocity V_1, then the line X_1 will represent, in direction and in magnitude, the velocity V_1. Similarly, if the end A remains fixed and the end B be moved in the direction BB' with such a speed that in unit time the point W traverses a distance X_2 proportional to the magnitude of the velocity V_2, then the line X_2 will represent, in direction and in magnitude, the velocity V_2. If now, both ends, A and B, of the rubber band be moved simultaneously with the same velocities as before, the paper wad W will, in unit time, move along the diagonal of the parallelogram constructed on X_1 and X_2 as sides, a distance WR proportional to the magnitude of the resultant of the two velocities V_1 and V_2.

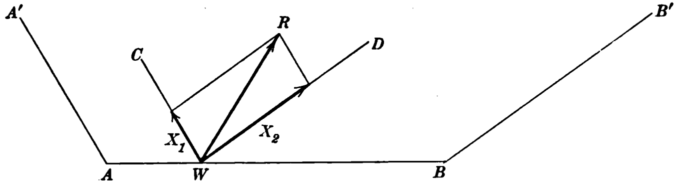

Fig. 44.

61. Since the parallelograms *ab* and *AB*, Fig. 43, are similar,

$$t' : t \; [= Oa : OA] = Oc : OC.$$

That is, the distance described along the resultant is directly proportional to the time. Therefore, *the resultant of two uniform velocities is a uniform velocity.*

We have just shown that if a man moves across a train in a direction *OA*, with a velocity ${}_{t}V_{m}=OA$, and if at the same time, the train is moving along the earth in some direction *OB* with a velocity ${}_{e}V_{t}=OB$, then the velocity of the man with reference to the earth, ${}_{e}V_{m}$ is represented by *OC*. In general, if one body moves with reference to a second, and at the same time the second moves with reference to a third, then the velocity of the first with reference to the third is represented in direction and in magnitude by the diagonal of the parallelogram constructed upon the lines representing the two component velocities, all three lines being drawn from the same point.

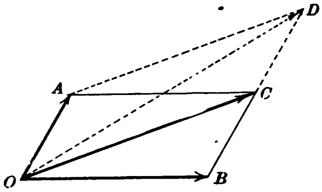

Fig. 45.

If, however, we attempt to combine in this way the velocity of the first in respect to the second with the velocity of the first in respect to the third, *e. g.* to combine the velocity of the man in respect to the train with the velocity of the man in respect to the earth, we should get a line *AD* which represents nothing that we already had on our diagram, and which certainly does not represent the velocity of the man with reference to either earth or train. In fact, this combination would be meaningless. In general, if we attempt to combine the velocity of the first in respect to the second, $({}_{2}V_{1})$, with any velocity except that of either the second in respect to something, $({}_{x}V_{2})$, or that of something in respect to the first, $({}_{1}V_{x})$, the result is meaningless. The only way in which it is permissible to compound velocities may be indicated as follows:—

${}_{2}V_{1}$ and ${}_{x}V_{2}$ give ${}_{x}V_{1}$,

and ${}_{2}V_{1}$ and ${}_{1}V_{x}$ give ${}_{2}V_{x}$,

If we adopt the notation used above, and write for any velocity a *V* with two subscripts, a right subscript to indicate the moving body and a left subscript to indicate the body to which the motion is referred, we have seen that two velocities can be compounded only when the right subscript of one is the same as the left subscript of the other. When we do so compound, the resultant velocity is the velocity of the body indicated by the remaining right subscript, and the body to which motion is referred is that indicated by the remaining left subscript.

62. Computation of the Resultant of two Uniform Linear Velocities.—The method of computing the resultant of two component velocities is the same as that used in Arts. 30 to 32, for the computation of the resultant of two concurrent forces. It is left to the student to show that if s represents the magnitude of the resultant of two component velocities of magnitudes s_1 and s_2, inclined to one another at an angle Φ, then

$$s^2 = s_1^2 + s_2^2 + 2\, s_1\, s_2 \cos \Phi. \tag{34}$$

Also, if the inclination of the resultant to the directions of s_1 and s_2 be denoted by Φ_1 and Φ_2, respectively, then

$$\sin \Phi_1 = \frac{s_2 \sin \Phi}{s} \tag{35}$$

$$\sin \Phi_2 = \frac{s_1 \sin \Phi}{s}. \tag{36}$$

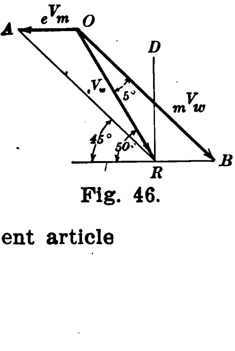

Fig. 46.

Problem.—To a man traveling west at 8 miles per hour the wind appears to come from the northwest. When he stands still, the wind appears to come from a point 5° more to the north. Find the speed of the wind.

Solution.—If the velocity of the man relative to the earth be denoted by the symbol ${}_eV_m$, the velocity of the wind relative to the man by ${}_mV_w$, and the velocity of the wind relative to the earth by ${}_eV_w$, then, from the law deduced in the present article

$${}_eV_m \text{ and } {}_mV_w \text{ give } {}_eV_w.$$

From the diagram,

$$\frac{{}_eV_w}{{}_eV_m} = \frac{OR}{OA} = \frac{\sin OAR}{\sin ORA}.$$

That is,
$${}_eV_w = \frac{8 \sin 45^\circ}{\sin 5^\circ} = 64.9 \text{ mi. per hr.}$$

63. Resolution of Uniform Linear Velocities.—If a resultant velocity is replaced by two component velocities, it is said to be resolved into these components. The remarks made in Arts. 36 and 37 with regard to the resolution of forces are also applicable to the resolution of velocities.

Fig. 47.

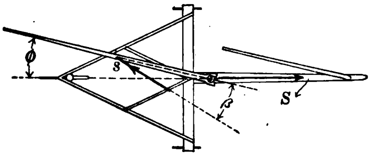

Fig. 48.

For example consider the resolution of linear velocities in the sailing of an ice yacht, (Fig. 47). Let the wind blow against the sail with a speed s at an angle β,(Fig. 48). Denote the angle between the sail and the line of the runners by Φ. Neglecting the slight friction of the runners on the ice, the speed S of the yacht will now be derived.

Since there is no frictional resistance to the yacht's motion, the speed will continue to increase so long as there is any wind pressure on the sail. There will be wind pressure on the sail until the component of the velocity of the yacht normal to the sail equals the component of the velocity of the wind normal to the sail. When this occurs, a particle of air will just slide along the sail without pressing on it. Under these conditions,

$$S \sin \Phi = s \sin \beta.$$

Therefore, if there is no retarding force due to friction between the ice and the runners, the speed of the yacht will be

$$S = \frac{s \sin \beta}{\sin \Phi}.$$

This equation shows that when β is less than Φ the speed of the yacht is less than that of the wind; when β equals Φ the speed of the yacht equals that of the wind; and when β is greater than Φ the speed of the yacht is greater than the wind which propels it.

PROBLEMS.

1.—A train A runs eastward at the rate of 45 mi. per hr. while another train B runs in the opposite direction at the rate of 35 mi. per hr. With what velocity will the train B appear to pass a passenger in A? [80 mi. per hour westward.]

2.—Two railroad tracks intersect at an angle of 90°. To a passenger on a train traveling east at the rate of 32 mi. per hr. the other train seems to have a velocity of 40 mi. per hr. south. Find the actual velocity of the second train. [23.98 mi. per hr., S. 53° 10′ W.]

3.—Two ships start at the same time from the same point, one sailing east at 12 mi. per hour, the other south at 9 mi. per hour. Find the velocity of the first relative to the second. [15 mi. per hr., N. 53° 10′ E.]

4.—A carriage is traveling at a rate of 10 mi. per hour through a rainstorm. The rain drops are falling vertically, but to a person in the moving carriage they appear to fall at an angle of 30° to the vertical. Find the speed of the descending raindrops. [17.31 mi. per hr.]

5.—A steamer is traveling north with a speed of 10 mi. per hr. in a wind blowing from the east at the rate of 10 mi. per hr. What direction will the smoke from the funnel appear to take as seen by a person on the steamer, and what will be its direction relative to the earth? Find also its speeds relative to the steamer and to the earth. [S. W.; W.; 14.14 mi. per hr.; 10 mi. per hr.]

6.—A steamer is moving due north with a speed of 25.6 km. per hour. The smoke from the funnel lies 35° south of east. If the wind is due west find its speed. [36.6 km. per hr.]

7.—A steamer is going west at the rate of 18 mi. per hr.; the current runs south 4 mi. per hr.; on shore is a train running west at 60 mi. per hr. Find the velocity of the train relative to the steamer. [22.4 mi. per hr., W. 10°20′ N.]

8.—While a train is moving with a speed of 40 mi. per hr., a man throws out a mail pouch in a direction normal to the track with a horizontal velocity of 20 ft. per sec. with respect to the car. Find the speed of the pouch with reference to the ground at the moment it leaves the man's hand. [42.25 mi. per. hr.]

9.—A boat is rowed west at the rate of 4 mi. per hr. across a river flowing south at 4 mi. per hr. A boy on the bank walks north at the rate of 3 mi. per hr. Find the relative velocity of the boy and the boat. [8.06 mi. per hr., 29°43′ to the bank.]

10.—A cannon ball fired from a ship has a horizontal velocity of 500 m. per sec. and strikes a target situated 2000 m. distant. If the ship be moving in a direction parallel to the target at a velocity of 25.2 m. per hr. find the distance between the point struck and the point aimed at. [0.028 m.]

11.—A river flows south at the rate of 5 mi. per hr. On it is a boat which if it were on still water would be going northeast at the rate of 8 mi. per hr. On the boat a flag is being hoisted at the rate of 2 ft. per sec. Find the speed of the flag with respect to the earth. [516 ft. per min.]

12.—When a train is moving with a speed of 20 mi. an hr. alongside a station platform, the guard throws out a parcel with a horizontal velocity of 16.9 ft. per sec., with respect to the train, in a direction normal to the track. Find the velocity of the parcel with respect to the earth. [33.8 mi. per hr. toward the rear of the train, at an angle of 30° with the track.]

13.—A railway train is moving at the rate of 28 mi. per hr. when a bullet moving horizontally enters a compartment at the corner furtherest from the engine and passes out at the diagonally opposite corner. If the compartment be 8 ft. long and 6 ft. wide, and the sine of the angle of inclination of the bullet's path to the train be 0.6, find the speed of the bullet. [80 mi. per hr.]

§ 2. *Uniform Angular Motion.*

64. Measurement of Angles.—The angle between two lines lying in the same plane is the difference in direction between them. In ordinary life the unit of angular measurement is arbitrarily taken as one nintieth part of a right angle. This unit is called the *degree*. In scientific work a unit of angular measurement called the *radian* is frequently employed. The radian is the angle subtended at the center of a circle by an arc equal to the radius of the circle.

Thus, in the figure

$$\frac{AB}{OA} = \Phi \text{ radians.}$$

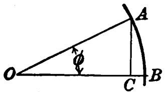

Fig. 49.

Again, from the definition,

$$360° \left[= \frac{2\pi r}{r} \right] = 2\,\pi \text{ radians.}$$

$$\therefore \quad 1 \text{ radian} = \frac{360°}{2\pi} = 57°\ 17'\ 44.8'' \text{ (nearly).}$$

The method of measuring angles which employs the radian as the unit is called *circular measure.*

65. When the angle Φ is small,

$$\frac{AB}{OA} = \frac{AC}{OA} \text{ (nearly).}$$

Consequently for small angles

$$\sin \Phi = \Phi \text{ radians (approx).}$$

For example,

$2° = 0.0349$ radians, and $\sin 2° = 0.0349$;

$4° = 0.0698$ radians, and $\sin 4° = 0.0698$;

$6° = 0.1047$ radians, and $\sin 6° = 0.1045$;

$8° = 0.1396$ radians, and $\sin 8° = 0.1392$.

66. Angular Velocity.—A displacement of a body such that all points of the body describe coaxial circular arcs is called *rotation*. The axis of rotation may pass through the body or it may be outside of the body. In rotation all lines perpendicular to the axis of rotation sweep through equal angles in equal times. Let XX' (Fig. 50) be a line fixed in space and let PO be a line fixed in the body normal to the axis of rotation passing through O. The rate of change of the angle XOP is called the *angular velocity* of the body about the assigned axis. The *angular speed* of the body is the magnitude of the rate of change of the angle XOP without reference to the axis about which rotation occurs. Thus, representing by w, the angular speed of the body,

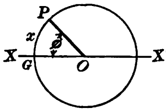

Fig. 50

$$w = \frac{\Phi}{t} \text{ radians per second.}$$

A *uniform angular velacity* is one having constant angular speed in a fixed direction about an invariable axis.

67. Representation of Angular Velocity.—Two quantities are required to specify completely an angular velocity—the angular speed, and the direction of the axis about which rotation occurs. An angular velocity can be completely represented by a straight line whose magnitude is proportional to the angular speed, and whose direction is parallel to the axis of rotation. Thus, the line AB, three units long, inclined as in the figure, represents an angular velocity of three radians per second about an axis parallel to AB. The sense of the rotation is indicated by the direction of the arrow-head. The arrowhead is so placed that on looking along the axis in the direction of the arrow, the rotation is clockwise.

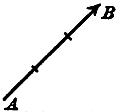

Fig. 51.

68. *Composition of Angular Velocities.—It will now be shown that two angular velocities, about axes that meet in a point, can be compounded in a similar manner to two linear velocities. Let OP and OQ be two axes about which a body rotates with angular speeds w_1 and w_2 respectively, in the directions indicated in the figure. About O as a center describe an arc of unit radius. From any point, C, of this arc, erect

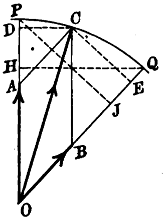

Fig. 52.

lines CD and CE perpendicular respectively to the two axes OP and OQ. Denote CD and CE by r_1 and r_2, respectively.

Due to the single angular velocity w_1 the line CD would rotate, about OP as an axis. in such a direction that the point C would be depressed below the plane of the paper. Denoting the angle swept through by the line CD in time t by the symbol Φ_1, we have

$$w_1 = \frac{\Phi_1}{t}.$$

Denoting by x_1 the distance traveled by the point C while CD sweeps through the angle Φ_1, we have

$$\Phi_1 = \frac{x_1}{CD} = \frac{x_1}{r_1}.$$

Consequently $$x_1 = \Phi_1 r_1 = w_1 r_1 t. \tag{37}$$

Again, due to the single angular velocity w_2, the line CE would rotate about OQ as an axis, in such a direction that the point C would rise above the plane of the paper. Denoting the angle swept through by CE in time t by the symbol Φ_2, and the distance traveled by the point C by x_2, we find in the same manner as above

$$x_2 = w_2 r_2 t.$$

The point C will be at rest throughout the time t if $x_1 = x_2$. But if the point C is at rest, the entire line OC will also be at rest; that is, OC must be the direction of the axis of the resultant angular velocity. Consequently, the line OC is the direction of the axis of the resultant angular velocity of w_1 and w_2 when

$$w_1 r_1 t = w_2 r_2 t,$$

or, $$\frac{w_1}{w_2} = \frac{r_2}{r_1}. \tag{38}$$

The condition required by this equation is easily determined. From C draw lines CA and CB parallel to the axes of w_1 and w_2, respectively. The area of the parallelogram $OACB$, thus constructed, is

$$(OA)\ r_1 = (OB)\ r_2.$$

Whence

$$\frac{OA}{OB} = \frac{r_2}{r_1}. \tag{39}$$

From (38) and (39) it is seen that the condition that the line OC shall be the direction of the axis of the resultant angular velocity of w_1 and w_2 is that

$$\frac{OA}{OB} = \frac{w_1}{w_2}.$$

It has now been proved that if the lines OA and OB are proportional to the angular velocities about these axes, then the diagonal of the parallelogram of which these are contiguous sides is the axis of the resultant velocity. It remains to show that the length of this diagonal is proportional to the magnitude of the resultant angular velocity.

Draw QH perpendicular to OP. Since Q is on the axis of w_2, the distance traveled by Q in time t is

$$w_1t(QH).$$

If the resultant angular velocity of the body about OC be denoted by w, we shall also have for the distance traveled by Q in time t, the value

$$wt(CE).$$

Whence

$$\frac{w_1}{w} = \frac{CE}{QH} = \frac{\sin COE}{\sin QOH} = \frac{\sin OCA}{\sin OAC} = \frac{OA}{OC}. \tag{40}$$

Similarly, by drawing PJ perpendicular to OQ, we find that

$$\frac{w_2}{w} = \frac{OB}{OC}. \tag{41}$$

Taken together, (40) and (41) show that, if lengths proportional to the respective angular velocities about them be measured off on the component and resultant axes, the lines so determined will be the sides and diagonal of a parallelogram.

It has now been proved that *if a parallelogram be constructed on two contiguous lines representing two angular velocities, the diagonal of the parallelogram will represent completely the resultant angular velocity.*

The composition of two uniform angular velocities about coplanar axes can be illustrated by means of a globe mounted as shown in Fig. 53. First, let the globe be set into uniform rotation in the clockwise direction, as seen from above, about the axis YY'. In unit time any point P will move to some point R with an angular velocity w_1, which can be represented by the line OA, (Fig. 54). If, instead of rotating about YY', the globe should rotate about the axis $X'X$ in the clockwise direction, as seen from the right, the point P would move in unit time to some point Q with an angular velocity w_2, which can be represented by the line OB, (Fig. 54). If both of these motions should occur at the same time, then the point P would be displaced to the point S, (Fig. 53), with an angular velocity w_r which the above theorem shows is completely represented by the line OC, (Fig. 54.)

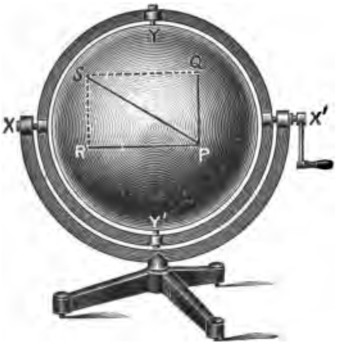

Fig. 53.

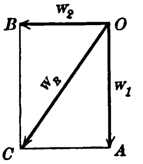

Fig. 54.

69. The Relation between Angular and Linear Speed.—In Fig. 50, imagine the body to rotate about O with an angular speed w. Then any point, P, fixed in the body, will move in the circumference of a circle of radius r with a linear speed s. Denoting the distance PG passed over in time t by x,

$$s = \frac{x}{t}.$$

If the angle passed over by PO in time t be called Φ,

$$w = \frac{\Phi}{t} = \frac{x}{r\,t} = \frac{s}{r}. \tag{42}$$

Whence, the angular speed of a body is numerically equal to the linear speed of any rotating point of the body divided by the distance of this point from the axis of rotation.

70. Instantaneous Axis of Rotation.—The fixed axis about which rotation occurs, is called the axis of rotation. Sometimes, however, this axis is fixed for but a very short space of time; it is then called the *instantaneous axis of rotation.*

If a wagon axle be lifted off the ground and the wheel set in rotation, the axis of rotation with reference to both the earth and the wagon will be permanent and at the center of the axle. If, however, the wheel rolls along the ground, the axis will not be at rest with reference to the earth, but will remain at rest with respect to the wagon. If the wheel is rolling along the ground without sliding, the point in contact with the ground is instantaneously at rest with respect to the ground. Therefore the instantaneous axis of rotation of a rolling wheel is at the point in contact with the ground. It must be remembered that at every succeeding instant this point is a different part of the wheel.

PROBLEMS.

1.—The earth makes a complete rotation in 76164 mean solar sec. Assuming the radius to be 6370900 m. find the angular speed, and also find the linear speed of a point on the equator. [0.0000824 radians per sec.; 524.962 meters per sec.]

2.—A carriage traveling at the rate of 7 mi. per hr. has a wheel 3 ft. in diameter. Find the angular speed of one of the spokes. [6.84 radians per sec.]

3.—Find the component angular velocity of the earth about a vertical axis passing through a place at latitude L. [$w = \frac{1}{12} \pi \sin L$ radians per hr.]

4.—Find the angular velocity of the minute hand of a clock relative to the hour hand. [5.7596 radians per hr.]

CHAPTER VI.

THE MOTION OF A BODY UNDER THE ACTION OF A CONSTANT FORCE.

§ 1. *Uniformly Accelerated Linear Motion.*

71. Linear Acceleration.—When a force acts upon a body, either the direction or the magnitude of the body's motion will tend to change during the time the force acts. If the velocity changes, either in direction or in magnitude, it is said to be *accelerated.* When the velocity of a body is accelerated, the *instantaneous velocity* at any point of its path equals the distance that would be traversed in one second, in the given direction, if from that point onward the velocity were to be uniform. If the velocity of a body changes at a uniform rate, the change in the linear velocity divided by the time occupied in producing the change is called the *linear acceleration* of the body's motion. Or, more briefly, linear acceleration is the time rate of change of linear velocity. Thus, if the magnitude of the velocity changes uniformly from s_o to s_t during the time t, then the acceleration a during this interval is

$$a = \frac{s_t - s_o}{t}. \tag{43}$$

This equation shows the unit of linear acceleration to be unit change of linear speed in unit time. The magnitude of the unit of linear acceleration used in science is the change of speed of one centimeter per second in one second; in engineering, the magnitude usually employed in English speaking countries is the change of speed of one foot per second in one second, or a change of speed of one mile per hour in one second.

If the speed diminishes with time, the acceleration is said to be *negative.* A negative acceleration is often termed a *deceleration* or *retardation.*

Nonuniform or accelerated motion is well illustrated by a heavy ball rolling down (or up) an inclined plane. A very satisfactory form of apparatus for class demonstration is shown in Fig. 55. This consists of a grooved plank about twenty feet long having at its upper end a device for starting the ball at any desired instant, and along the length of the groove a series of incandescent lamps that flash up, one after the other, at one second intervals. Simultaneously with the release of the ball at the top of the incline, the lamp 0 flashes up, one second later the lamp 1 flashes up, two seconds after the ball starts the lamp 2 flashes up, and so on. The position of the lamps along the grooved plank can be adjusted so that the rolling ball will be exactly in front of each lamp at the instant it flashes. The distance between any two consecutive lamps will then be the distance traveled by the ball during the corresponding second of time.

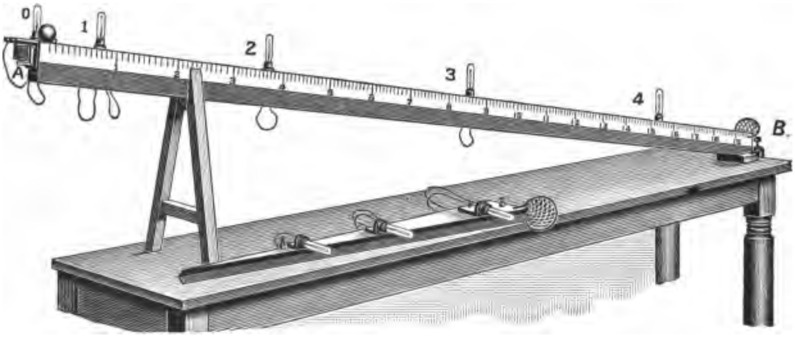

Fig. 55.

In a particular trial, the ball traveled one foot during the first second, three feet during the second equal interval of time, five feet during the third second, seven feet during the third second. Since unequal distances were traversed during equal intervals of time, the velocity was not consant but was accelerated. Since the speed increased with time, the acceleration was positive.

From the definition of linear acceleration, if the speed of a body changes from s_0 to s_t during the time t, the mean acceleration during this interval is

$$a = \frac{s_t - s_0}{t}.$$

In order to determine, by means of this equation, the magnitude of the linear acceleration of the ball's motion during this interval, it will be necessary to know the instantaneous speed of the ball at the beginning and end of the interval. The instantaneous speed of a body at any instant equals the distance that would be traversed in one second, if from that instant the speed were to be uniform. Since there is no horizontal component of the weight of a body, the law of inertia (Art. 10) shows that a body moving along a smooth horizontal surface, and unacted upon by any force except its weight, will move with a velocity constant both in direction and magnitude. Consequently, if the ball be intercepted at any point of its path down the inclined plank by a horizontal track, the instantaneous speed of the ball at the given point can easily be measured. In the present apparatus this is accomplished by means of the wedge shaped piece *W*. In Fig. 56 this is shown in position for finding the instantaneous speed of the ball at the end of the first second after starting from rest.

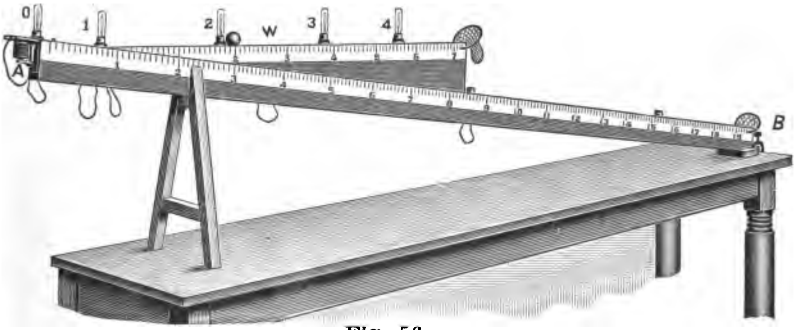

Fig. 56.

When the wedge was in this position, the ball traveled along the horizontal grooved plank two feet in one second. When the acute angle of the wedge was placed at the positions reached by the ball in its motion down the incline at the end of the second, third, and fourth seconds, the intercepted ball was found to travel on the horizontal track in one second, four, six, and eight feet respectively. Whence, the instantaneous speed of the ball down the inclined plank at the end of the first, second, third, and fourth seconds was two, four, six, and eight feet per second, respectively. It thus appears that during any one-second interval of time the speed changed two feet per second, or there was a constant acceleration down the plank of two feet per second in one second. These results may be arranged as in the following table.

Time	*Total distance*	*Speed at end of each sec.*	*Acceleration.*
0 sec.	0 ft.	0 ft. per sec.	
1 "	1 "	2 "	2 ft. per sec. per sec.
2 "	4 "	4 "	2 "
3 "	9 "	6 "	2 "
4 "	16 "	8 "	2 "

If instead of being straight, the upper edge of the inclined plank had been of a curved form, the acceleration of the ball's motion would have been ununiform. Depending upon the inclination of the groove, the acceleration might be positive, zero, or negative at different instants of time.

It is one of the fundamental principles of dynamics, (Art. 11), derived from experience and experiment, that when a force acts upon a body, the acceleration thereby produced is in the line of action of the force, and in magnitude is directly proportional to the force applied. From this principle it follows that if a constant force be applied to a body a uniform acceleration will be produced.

A uniform or an instantaneous linear acceleration is completely described when both its direction and its magnitude are specified. Consequently, a linear acceleration can be completely represented by a straight line. Linear accelerations can be compounded and resolved by the same methods used for forces and velocities.

72. Acceleration due to Gravity.—Newton, Galileo, and others have proven experimentally that at any point of the earth, terrestrial gravitation imparts to all bodies equal accelerations, but that at different places, it imparts to the same body different accelerations. The acceleration due to gravity depends upon the latitude of the place, its distance from the center and from the axis of the earth, together with such local conditions as the presence of mountains, large deposits of metals, etc. At sea level, at the equator, the acceleration due to gravity is 978 cm. per sec. per sec. (32.09 ft. per sec. per sec.), while at the pole it is 983 cm. per sec. per sec. (32.26 ft. per sec. per sec.). The acceleration due to gravity is usually represented by the symbol g.

A body on the inclined plane AB is under the influence of the force of gravity vertically downward. If the acceleration in the direction of this force be called g, the component parallel to a plane inclined at the angle Φ with the horizon is directed down the plane and has the value $(g \sin \Phi)$.

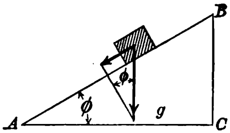

Fig. 57.

If a body starting at A with a certain initial velocity were to ascend the smooth plane AB, its velocity would experience an acceleration in the direction of the motion equal to $(-g \sin \Phi)$.
As Φ diminishes, the magnitude of this retardation decreases, until in the limit when $\Phi = 0$, the retardation is zero. Consequently, if there were no friction or other resistance, a body would move along a level surface with a velocity constant both in direction and in magnitude.

73. Acceleration produced by a Uniform Force acting in the Direction of Motion.—If a uniform force acts upon a body in the direction of its motion, the speed of the body will change at a uniform rate but the direction of motion will remain unaltered.

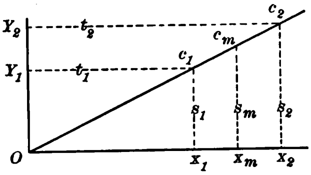

Fig. 58.

Howsoever the speed of a body may change during a given time, there must be a certain equivalent speed with which a uniformly moving body would traverse the same distance in the same time. The value of this equivalent speed for the case of a body moving with uniform linear acceleration will now be determined. Let OX_1 and OX_2, (Fig. 58), represent intervals of time t_1 and t_2 measured from some arbitrarily selected starting point O; and let OY_1 and OY_2 represent the instantaneous speeds s_1 and s_2 of the body at these particular instants. Draw X_1c_1 and X_2c_2 parallel to the axis of speeds, Y_1c_1 and Y_2c_2 parallel to the axis of times and draw the lines Oc_1 and Oc_2. Now since the change of speed during the time t_1 is represented by X_1c_1, and the change of speed during the time t_2 is represented by X_2c_2, and since the motion being considered is uniformly accelerated, *i. e.* is motion in which the change of speed is proportional to the time which elapses, the acceleration is given by

$$a = \frac{X_1c_1}{OX_1} = \frac{X_2c_2}{OX_2}.$$

Consequently the triangles OX_1c_1 and OX_2c_2 are similar, and the sides Oc_1 and Oc_2 are coincident. Similarly it may be shown that the extremity of the ordinate representing the speed of the body at any instant must lie on the line Oc_2. If s_m is the instantaneous speed at any time t_m, then the short distance δd traversed during a very short time interval δt just before or just after t_m is very nearly $s_m\delta t$, and the shorter δt the more nearly does this product represent the distance δd. But the product also represents on the diagram a narrow area of width δt lying close to s_m and extending from X_m to c_m. The whole distance traveled during the time $(t_2 - t_1)$ is the sum of the distances traveled during the elementary time intervals into which $(t_2 - t_1)$ may be divided, and the sum of the elementary areas which represent on the diagram the distances traveled during the elementary time intervals is the area of the trapezoid c_1X_2. Thus the area of this trapezoid represents the distance traversed by the body during the time $(t_2 - t_1)$.

If this distance is divided by the time $(t_2 - t_1)$, the quotient will be the uniform speed which a body would have to possess in order to traverse the distance represented by c_1X_2 in the time $(t_2 - t_1)$. But the quotient of the area c_1X_2 by the distance X_1X_2 is the ordinate half way from X_1 to X_2. It follows that the equivalent uniform speed of the body during the interval from t_1 to t_2 is given by the ordinate of the point midway between c_1 and c_2. In other words, if the motion is uniformly accelerated, the equivalent uniform speed equals the arithmetical mean of the instantaneous speeds at the beginning and end of the time considered. Or, if during a time t the speed changes from s_o to s_t at a uniform rate, then the mean equivalent speed during this interval is

$$s_e = \tfrac{1}{2}\,(s_o + s_t)\;t.$$

Therefore, the distance x traveled during the time t by a body whose speed during that time changes uniformly from s_o to s_t, is

$$x\;[= s_e\,t] = \tfrac{1}{2}\,(s_o + s_t)\;t. \qquad (44)$$

From the combination of (43) and (44) various important equations of uniformly accelerated linear motion can be obtained. In deriving from these equations a formula for the solution of any particular problem, the proper quantity to be eliminated can be determined from an inspection of the data furnished by the problem.

Problem.—While running at 45 miles per hour the brakes are applied to an electric car in such a manner as to bring the car to rest in 500 feet. A man is standing in the middle of the track 500 feet in front of the car when the brakes are applied. If in order to escape being touched by the car the man must walk 5 feet at right angles to the track, how close can he let the car come to him before starting to walk at the rate of 3 miles per hour?

Solution.—First the acceleration of the car's motion will be found. Combining (43) and (44) by eliminating t, and remembering that in this problem the final speed $s_t = 0$, we have

$$a = -\frac{s_0^2}{2x}.$$

Now,

$$s_0 = 45 \text{ mi. per hr.} = \frac{45\ (5280) \text{ ft.}}{60\ (60) \text{ sec.}} = 66 \text{ ft. per sec.}$$

So that

$$a = -\frac{(66)^2}{2(500)} = -4.36 \text{ ft. per sec. per sec.}$$

The time required by the man to walk five feet at the rate of 3 miles per hour will next be computed.

$$3 \text{ mi. per hour} = \frac{3(5280) \text{ ft.}}{3600 \text{ sec.}} = 4.4 \text{ ft. per sec.}$$

Whence, the time required by the man to walk 5 ft. at this rate is $5 \div 4.4 = 1.136$ secs.

The problem now resolves itself into finding the distance the car moves during the 1.136 secs. immediately before coming to rest. Eliminating s_0 from (43) and (44), and remembering that $s_t = 0$, we get

$$x = -\tfrac{1}{2} at^2 = \tfrac{1}{2} .\ 4.36\ (1.136)^2 = 2.81 \text{ ft.}$$

74. Acceleration produced by a Uniform Force acting perpendicularly to the Direction of Motion.—Since the acceleration produced is always in the direction of the applied force, if a force acts upon a body in a line perpendicular to the direction of its motion, the speed of the body will remain unaltered. An acceleration perpendicular to the direction of the motion signifies that the path of the body is changed by the force.

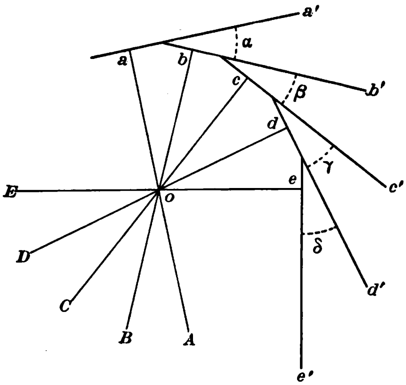

Fig. 59.

Consider a body which, at some given instant, is at a, Fig. 59), moving in the direction aa', and acted upon by a force which has the direction aA, normal to aa' and in the plane of the paper. This force will cause the body to change the direction of its motion so that after a short time interval t the body will be at some point b and moving in a direction bb'. At b the force acts in the direction bB normal to bb' and in the plane of the paper. After successive equal intervals t, the body will be successively at c, d, e, etc., moving in directions cc', dd', ee', etc., and the force will be acting successively along cC, dD, eE, etc. Since the body is moving with uniform speed, the lengths of the curved paths which it moves during the successive equal intervals t are equal. That is

$$\text{arc }(ab) = \text{arc }(bc) = \text{arc }(cd) = \text{etc.} \qquad (45)$$

Moreover, since the force acting is constant in magnitude, the changes in direction which it produces during equal time intervals are equal. That is

$$\angle\, \alpha = \angle\, \beta = \angle\, \gamma = \text{etc.} \qquad (46)$$

Equations (45) and (46), taken together, show that the points a, b, c, etc., are on a line of constant curvature.

Now it is shown in pure mathematics that the only plane curve of constant curvature is the circle. Consequently, the points a, b, c, etc., are all in the circumference of a circle. In the limit, when t is indefinitely small, a, b, c, etc., are consecutive points in the circumference of a circle. Therefore, a body moving with uniform speed, and acted upon by a constant force always perpendicular to the direction of motion, describes the circumference of a circle. The acceleration of the body's motion is directed toward the center of the circular path.

The magnitude of this acceleration will now be determined. Let the body move with uniform speed s in a circle of radius r and center O. Assume that in the short interval of time t the body has traversed the distance bc. Then

$$\text{arc }(bc) = st. \qquad (47)$$

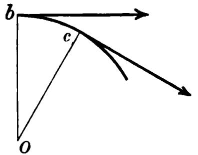

Fig. 60.

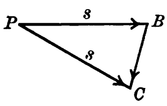

Fig. 61.

From a point P, draw PB and PC representing in direction and magnitude the velocity of the body when in the positions b and c, respectively. From the principle of the composition of velocities it follows that the line BC represents, in direction and in magnitude, the change in the velocity of the body during the time t. Whence, denoting the acceleration by a,

$$(BC) = at. \tag{48}$$

Dividing (48) by (47)

$$\frac{(BC)}{\text{arc } (bc)} = \frac{a}{s}. \tag{49}$$

But in the limit, when t is indefinitely small, bOc and BPC are similar triangles. Hence,

$$\frac{(BC)}{\text{arc } (bc)} = \frac{(PB)}{(Ob)} = \frac{s}{r}. \tag{50}$$

From (49) and (50) it is seen that the acceleration of a body moving with constant speed and acted upon by a force always normal to the direction of its motion is

$$a = \frac{s^2}{r}. \tag{51}$$

It has now been proved that *when a body is acted upon by a force of constant magnitude whose line of action is always perpendicular to the direction of the motion the body will describe with uniform speed a circular path. The motion of the body will have an acceleration directed toward the center of the circular path, having a constant magnitude equal to the square of the linear speed of the body divided by the radius of its path.*

The converse of this proposition may be stated as follows: *Whenever a body moves with constant speed in the circumference of a circle, there is acting upon the body at every point of its path a force directed toward the center. This force produces an acceleration directed toward the center of the circular path, equal to the square of the linear speed of the body divided by the radius of its path.*

75. If the body makes n revolutions per second,

$$a = \frac{s^2}{r} = \frac{(2\pi r n)^2}{r} = 4\pi^2 r n^2. \tag{52}$$

If the time of revolution is T seconds,

$$a = \frac{s^2}{r} = \frac{1}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 r}{T^2}. \tag{53}$$

If the angular speed be represented by w,

$$a = \frac{s^2}{r} = \frac{(w r)^2}{r} = w^2 r. \tag{54}$$

PROBLEMS.

1.—A train is moving on a level track with a speed of 40 mi. per hr. If the brakes are applied so as to produce a retardation in the speed of 10 ft. per min. in a sec. find the time occupied in bringing the train to rest. [5.8 min.]

2.—When the U. S. "Raritan" was launched she was observed to start from rest and slide down the ways through a distance of 54 ft. in 11 sec. The inclination of the ways to the horizontal was 3°40′. Find the coefficient of friction. [0.035.]

3.—A coaster after coming down a hill runs up another hill a distance of 200 ft. from its foot in 10 sec. when the sled stops. If the slope of the hill be 6° find the coefficient of friction. [0.0209.]

4.—A boy on a bridge wishes to drop a stone into the center of the smokestack of a locomotive passing underneath with a speed of 40 mi. per hr. If the level of the boy's hand is 60 ft. above the top of the smokestack, what must be the horizontal distance from the smokestack to the boy when the stone leaves his hand? [111.34 ft.]

5.—Show that the speed acquired by a body is sliding down a frictionless plane is the same as that acquired in falling freely through a vertical distance equal to the altitude of the plane.

6.—A ball fired horizontally from a gun placed 3 m. above the sea is seen to strike the water at a point 1000 m. distant. Find the initial velocity of the ball in the horizontal direction. [1282 m. per sec.]

7.—A cannon 40 m. above a plane discharges a ball with a velocity of 100 m. per second horizontally. Find the position of the ball at the end of the first and second seconds, the time the ball will remain in the air, the horizontal distance traversed. [In one sec. ball falls 4.9 m. and moves horizontally 100 m. In two seconds ball falls 19.6 m. and moves horizontally 200 m. Ball remains in the air 2.8 sec. Total horizontal distance traversed is 280 m.]

8.—A freight car with brakes set is dropped from the end of a moving train, the train meanwhile proceeding with uniform speed. The detached car comes to rest under the action of a uniform retarding force which produces a uniform negative acceleration. Prove that when the car stops, the distance of the train in front of it equals the distance through which the car has traveled from the instant of being detached.

9.—A brakeman on the roof of a freight car moving at the rate of 25 mi. per hr. throws a stone vertically upward with an initial velocity of 60 ft. per sec. Assuming that the air offers no resistance to the motion of the stone, find its velocity at the end of one second. Also find the distance from the starting point to the point where the stone falls. [27.9 ft. per sec. upward; 132.8 ft.]

10.—Find the gradient of a railway such that a carriage descending it by its own weight may travel 400 m. during the first minute. Find also how far the carriage will go during the second minute. Friction is neglected. [1° 20′; 1222.8 m.]

§ 2. *The Absolute Measure of Force.*

76. Mass and Inertia.—If an iron rod be moved from the equator to the pole of the earth, or be carried from the bottom of a mine to the top of a mountain, it will be found that its weight and some other of its physical properties will be altered, but during these changes of position there is one property that does not change. If the iron rod be distorted, electrified, heated, or magnetized, its energy, its volume, and certain other properties will be modified, but during all of these changes there is one property that remains unchanged. If it be dissolved in an acid, it will disappear as metallic iron and be transformed into an entirely different substance. Still, during this transformation there is one thing that does not change. That physical "something" which remains invariable however the position of a body may be changed, however its mechanical, electrical, thermal, or magnetic condition may be altered, or however its chemical constitution may be transformed, is called *mass.*

That characteristic of matter by virtue of which a body tends to persist in its present state of motion is called *inertia.* If the action of a given force upon each of two bodies, B_1 and B_2, gives to them the respective accelerations a and na, then the inertia of B_1 is said to be n times as great as that of B_2. The inertia of a body is uninfluenced by changes of position or by physical or chemical transformations. Since mass and inertia are so intimately connected, the inertia of a body is taken as a measure of its mass. That is, two masses can be compared in terms of their inertiae.

The unit of mass is arbitrarily taken to be either the mass of a certain piece of metal deposited in the Office of the Exchequer in London, or the mass of another piece of metal preserved in the Archives of Paris. The mass of the first is called one *pound,* and the mass of the second is called one *kilogram.*

In scientific work, the one thousandth part of the kilogram is the unit of mass usually employed. This is called the *gram.*

It was originally intended that the gram should be the mass of one cubic centimeter of water at its temperature of maximum density, *i. e.* at 4° C. or 39.°2 F. The material standard in Paris does not exactly realize this intention, but the departure is so slight that it is quite negligible except in the most refined work.

77. *Comparison of Masses.—Masses can be compared in a variety of ways. For instance, the masses of two bodies can be compared by determining the ratio of the linear accelerations produced in their motion by the application of equal forces. The magnitude of this force need not be known.

Consider two bodies *A* and *B* (Fig. 62) through which passes a smooth rod capable of rotation in a horizontal plane by means of the vertical spindle *C*. Let the two bodies be connected by a light cord. When the apparatus is rotated, *A* and *B* will tend to move away from the axis of rotation. A position can be found, however, at which they will remain in equilibrium;—that is, at which the forces acting on the two bodies due to the rotation are equal.

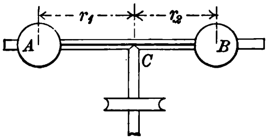

Fig. 62.

If the apparatus makes *n* revolutions per second, the acceleration of the body *A*, due to the rotation, is, from (52),

$$a_1 = 4\,\pi^2 r_1 n^2,$$

and for the body *B*, the acceleration due to the rotation is

$$a_2 = 4\,\pi^2 r_2 n^2,$$

where r_1 and r_2 are the distances of *A* and *B*, respectively, from the axis of rotation when the rotating system is in equilibrium.

Since the two bodies are acted upon by the same force, it follows, from the definitions of the previous article, that the accelerations of the two bodies are inversely proportional to their masses. That is, if the masses of *A* and *B* be denoted by m_1 and m_2 respectively,

$$\frac{m_1}{m_2} = \frac{a_2}{a_1} = \frac{r_2}{r_1}.$$

Whence, the masses of two bodies can be compared in terms of two accelerations, or in terms of two more easily measured distances.

There are other methods of comparing masses that offer less experimental difficulty than the one just described, but this one has been given in order to show that the essential nature of mass is quite distinct from that of force, and that masses can be compared without any knowledge regarding the magnitude of any force.

78. The Kinetic Units of Force.—The units of force employed in this course thus far have been the weights of certain arbitrarily selected bodies when at definite places on the earth's surface (Art. 9). Since the weight of a body does not change greatly when it is moved from one place on the earth's surface to another, these gravitational units of force, such as the pound weight and the kilogram weight, are usually used roughly as the weight of a mass of one pound or one kilogram at any place. This is convenient and sufficiently accurate for the requirements of engineering and ordinary life, but for scientific work it is necessary to have a unit that is absolutely constant.

The kinetic unit of force is based on the relation between force and the acceleration that it produces in a body's motion. From Newton's second law of motion, when a force acts upon a body an acceleration is produced in the line of action of the force, of a magnitude directly proportional to the applied force. That is,

$$a \propto F.$$

In Art. 76 we have seen that if equal forces act upon two bodies, the accelerations produced are inversely proportional to the masses of the bodies. That is,

$$a \propto \frac{1}{m}\cdot$$

Since no factor except force and mass affects the acceleration of a body's motion, it follows that (¹)

$$a = k_1 \frac{F}{m}\cdot$$

or

$$F = k(ma), \tag{55}$$

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(1). "If A varies jointly as B and C, then $A = k\,BC$."—Hall & Knight, Higher Algebra, p. 33.

where k is a constant of proportionality depending only upon the magnitude of the units adopted. If the unit of force be defined as that force which acting on unit mass will impart to it unit acceleration, then k becomes unity. In this case

$$F = m a. \tag{56}$$

If the units of length, mass, and time, be the centimeter, gram, and second, respectively, then the absolute unit of force is that force which will give to a mass of one gram an acceleration of one centimeter per second in one second. This unit of force is called the *dyne*. If, however, the units of length, mass, and time, be the foot, pound, and second, then the absolute unit of force would be that force which will give to a mass of one pound an acceleration of one foot per second in one second. This unit of force is called the *poundal*.

The resultant force acting on a body moving with uniform acceleration is sometimes called the *force of inertia* or *mass-acceleration.*

79. Weight proportional to Mass.—The weight of a body is the force exerted upon it by the attraction of the earth. Denoting by f_1 and f_2 the weights of two bodies of masses m_1 and m_2, we have at a place where the acceleration due to gravitation is g,

$$f_1 = m_1 g \qquad \text{and} \qquad f_2 = m_2 g.$$

Whence,

$$f_1 : f_2 = m_1 : m_2.$$

Consequently, *at any assigned point on the earth, the masses of two bodies are proportional to their weights at that point.* This law is the basis of the ordinary method of comparing masses, called weighing.

There are two sorts of balances in common use for comparing the masses of bodies—the beam balance and the spring balance. The ordinary beam balance consists of two similar scale pans suspended from the ends of a horizontal beam resting at its middle point on a fixed knife edge. In using the beam balance, the object to be weighed is placed on one scale pan, while on the other scale pan are placed standard masses (often inaccurately called "weights") until the beam is in equilibrium. Then

$$m_1 g = m_2 g.$$

This equation shows that the result is quite independent of the acceleration due to gravity at the place where the weighing is performed.

The spring balance consists of a spiral wire which is distorted by the application of a force. The spring balance can be calibrated by noting the elongations of the spring produced by a series of different standard masses. But since the elongation of the spring is due to the weight of the applied mass, and since the weight of a body depends upon the acceleration due to gravity, it follows that a spring balance calibrated for one place on the earth will give erroneous indications at any place where the acceleration due to gravity is different.

Problem.—Find the magnitude of the constant horizontal force required to stop, in 500 ft., a train of 300 tons (mass) running at 45 miles per hour.

Solution.—As in the solved problem of Art. 73, it can be found that if a train running at 45 mi. per hr. is brought to a stop in 500 ft., the acceleration is

$$a = -4.36 \text{ ft. per sec. per sec.}$$

Therefore, from (56), the force required to impart to the train this acceleration is given by the equation

$$F = ma = -(300 \times 2000 \times 4.36) = -2{,}616{,}000 \text{ poundals}$$
$$= -(2{,}616{,}000 \div 32.1) \text{ pounds' weight.}$$

The negative sign signifies that the direction of the force is opposite to the direction of motion.

80. Change of Apparent Weight due to Acceleration.—When a man is in an elevator which is either ascending or descending with uniform speed, the reaction of the floor on the man equals the man's weight. But if the elevator is ascending with an accelerated motion it has not only to support his weight but must also impart an acceleration to his motion. In this case, the reaction of the floor equals the man's weight plus the force required to give him the upward acceleration of the elevator. If his mass be m, and the acceleration due to gravity be g his weight is mg. If the upward acceleration of the elevator be A_u, then the reaction of the floor on the man, or his *apparent* weight, is

$$R = mg + mA_u$$
$$= mg\left(1 + \frac{A_u}{g}\right).$$

Hence the weight of the man has apparently increased by the fraction $A_u \div g$ of his real weight. If he were suspended by a spring balance it would indicate the weight given above.

If, on the other hand, the elevator were descending with an acceleration A_d, then the man's thrust on the floor would be less than his weight by the amount necessary to impart to him the downward acceleration of the elevator. Thus, his apparent weight is now

$$R' = mg - mA_d$$
$$= mg\left(1 - \frac{A_d}{g}\right).$$

If $A_d = g$, that is, if the elevator were falling freely, the man would exert zero pressure on the floor of the elevator.

81. D'Alembert's Principle.—A body of small mass and negligible dimensions is called a *particle*. A material system may be imagined to consist of an indefinitely large number of particles between which there are internal forces or stresses. From the principle of action and reaction, the stresses between any two bodies or particles of a material system consist of pairs of equal and oppositely directed forces. Consequently, *the internal forces existing between any system of bodies or particles constitute a system of forces in equilibrium.* This statement, however, must not be understood to say that the internal forces acting upon any one particle in the system are in equilibrium. That may or may not be true.

The force which, if it should act at any instant upon an isolated part of a system would cause it to move at that instant precisely as it actually does move while constituting a part of the system is called the *effective* force acting upon the body or particle. The effective force acting upon any particle is, then, nothing more nor less than the resultant of all the forces, external and internal, that are acting upon that particle. Since the equilibrant of any system of forces is equal and opposite to the resultant of the same system of forces, it follows that if, in addition to all the forces that are actually acting upon the various particles of any system, there could be called into existance a set of forces, each of which should act upon one particle of the system and should be equal and opposite to the effective force acting upon that particle, then every particle in the system would be in equilibrium. If a force equal and opposite to an effective force is called a reversed effective force, it follows, since the resultant of all the internal forces acting upon a system is zero, that *the external forces acting upon any system, and the reversed effective forces of that system, together constitute a system of forces in equilibrium.* This is called d'Alembert's Principle. If the resultant of all the external forces is denoted by F_{ex} and the resultant of all the effective forces by F_{eff}, the principle may be stated analytically by the equation

$$F_{ex} - F_{eff} = 0. \qquad (57)$$

It must be kept clearly in mind that the so called "effective" force is purely hypothetical and fictitious. It is assumed to act in the direction of the resultant acceleration. The "reversed effective" force is in the opposite direction. The great importance of D'Alembert's principle is due to the fact that by it a large class of problems involving accelerated motion can be reduced to problems of equilibrium and their solution be thereby greatly simplified.

82. Density and Specific Gravity.—The *density* of a substance is the ratio of its mass to its volume. Its magnitude depends upon the units in which mass and volume are measured. For instance the density of copper is 8.92 grams per cubic centimeter, or 557 pounds per cubic foot.

The relative density, or *specific gravity* of a substance is the ratio of its density to the density of some standard substance. In other words, the specific gravity of a body is the ratio of its mass to the mass of an equal volume of a standard substance. Specific gravity is an abstract number which is independent of the units employed. In the case of solids and liquids, water at the temperature of its maximum density (4° C. or 39.°2 F.) is arbitrarily taken as the substance with which the densities of other substances are compared.

Since the gram is the mass of one cubic centimeter of water at the temperature of its maximum density, it follows that the density of a body in grams per cubic centimeter is numerically equal to its specific gravity.

PROBLEMS.

1.—An arctic explorer uses a spring balance which was graduated at the equator. Find the greatest correction he may require to apply to his readings. Take the value of g at the equator and at the pole to be 32.09 and 32.25 ft. per sec. per sec. respectively. [— 0.49%.]

2.—An elevator starts to descend with an acceleration of 3 m. per sec. in a sec. Find the thrust on the floor produced by a mass weighing 75 kg. Find the thrust when the elevator starts to ascend with the same acceleration. [52.04 kg. wt.; 97.96 kg. wt.]

3.—A mass of 40 kg. is placed on a plane inclined 30° to the horizontal. The coefficient of friction is 0.3. What force will be required to give the body an acceleration of 300 cm. per sec. in a sec. up the plane? [(41.78) 10^6 dynes.]

4.—A man in the car of an elevator holds a 10 lb. package in his hand. The elevator starts to rise with a uniform acceleration such that the car is raised 10 ft. in 2 sec. The cable then breaks, thus allowing the car to fall freely to the bottom of the shaft. Find the pressure of the package on the man's hand when the elevator was rising and also when it was falling. [11.8 lb. wt. ; 0.]

5.—A body is suspended by a string from the ceiling of a railway car. When the train starts the string is deflected 10° from the vertical. Find the acceleration of the train on starting. [5.63 ft. per sec. per sec.]

6.—An engine winds a cage weighing 3000 kg. up a shaft at a uniform speed of 10 m. per sec. Find the tension of the rope. Find also the tension if the cage rise with a uniform acceleration of 10 m. per sec. in a sec. [3000 kg. wt.; 6061 kg. wt.]

7.—What is the force equivalent to the weight of 3 kg. at a place where a body starting from rest falls freely through 44.1 m. in 3 sec? [2,940,000 dynes.]

8.—A force of 4 lb. wt. causes a certain mass to move from rest through 18 ft. in 3 sec. Find the mass. [32 lb.]

9.—A body moves 10 ft. along a horizontal plane before coming to rest. If the coefficient of friction be 0.1, find the initial speed of the body. [8 ft. per sec.]

10.—A body is projected along a horizontal plane with a speed of 100 ft. per sec. If the coefficient of friction is 0.1, find how far the body will move. [1562.5 ft.]

11.—A body starting from rest slides down an inclined plane rising 3 ft. in 5. If the coefficient of friction is 0.2, find the speed gained in traversing 20 ft. of the plane. [220.6 ft. per sec.]

12.—A spring balance fastened to the roof of a moving elevator car indicates 75 lb. as the weight of a 100 lb. mass. Find the acceleration of the motion of the car. [8 ft. per sec. per sec.]

13.—What constant horizontal force is required to stop a train of 100 tons mass running at 50 mi. per hr., (a), in one min., (b), in 200 yd.? [3.8 tons wt.; 1.3 tons wt.]

14.—A force equal to the weight of one ounce acts upon a pound for 10 sec. Find the speed generated and the distance through which the mass will be moved in 10 sec. if the body starts from rest. [20 ft. per sec. ; 100 ft.]

15.—A car of 80 tons starting from rest on a level road, has a speed of 30 mi. per hr. at the end of the first mile. Determine the average tractive force of the engine, (a), if there were no frictional resistances, (b), if there are frictional resistances of 8 lb. wt. per ton. Also find what tractive force is required to haul the same car over a level road at constant speed. [915 lb. wt.; 1555 lb. wt.; 640 lb. wt.]

§ 3. *Momentum.*

83. Linear Momentum.—Let a system of free bodies of masses m_1, m_2, etc., be acted upon by a set of both external and internal forces. Let the resolved parts of the external forces acting on the respective bodies in any assigned direction be F_1, F_2, etc. Let the components, in the assigned direction, of the internal forces acting on the system due to the mutual action of its various parts, be denoted by f_1, $-f_1$, f_2, $-f_2$, etc. During the interval of time t, the various bodies will suffer a change of velocity, in the assigned direction, which will be denoted by Δv_1, Δv_2, etc.

Then, the sum of the forces acting in the given direction is

$$F_1+F_2+\text{ etc. }+f_1+f_2+\text{ etc. }-f_1-f_2-\text{ etc.}=\frac{m_1\,\Delta v_1}{t}+\frac{m_2\,\Delta v_2}{t}\cdot\cdot+$$

Consequently,

$$F_1\,t+F_2\,t+\text{etc.}=m_1\,\Delta v_1+m_2\,\Delta v_2+\text{etc.}$$

Representing the resultant force in the assigned direction by F,

$$F\,t=\Sigma\,(m\,\Delta v\,). \qquad (58)$$

The product of the force acting upon a body, and the time of its action, (Ft), is called the *impulse* of the force. The product of the mass of a body, and its linear velocity, (mv), is called the *linear momentum* of the body in the direction of motion.

The above equation shows that *the change in the momentum of a free material system, in any assigned direction, is determined solely by the external impulses acting upon the system.*

84. Conservation of Linear Momentum.—In the special case in which a system of free bodies is acted upon by external forces the sum of whose components in any assigned direction is zero, we have

$$F=\Sigma\,ma=\frac{\Sigma\,(m\,\Delta\,v)}{t}=0.$$

That is, in this special case, the rate of change of momentum is zero—in other words, the momentum is constant.

Therefore, *if a free material system is acted upon by a set of external forces whose resultant in any assigned direction is zero, the momentum of the system in the assigned direction is a constant quantity.* This theorem is called the Principle of the Conservation of Linear Momentum.

The recoil of a rifle as the bullet leaves it, is a familiar example of the conservation of linear momentum. Let m_1 and m_2 be the masses of the rifle and of the ball, respectively. Let v_1 and v_2 represent the magnitudes of the velocity of the rifle and of the ball, before firing, and let v_1' and v_2' represent their magnitudes after firing.

Since the momentum of the system is constant, the momentum before firing equals that after firing. That is

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'.$$

Before the ball was fired, the velocity of both the rifle and ball were zero: consequently at that time, the momentum of the system was zero. Therefore,

$$m_1 v_1' + m_2 v_2' = 0.$$

From this equation, knowing the masses of the rifle and the ball, together with the velocity of one of them at any given moment, the velocity of the other at that moment can be computed.

Again, if we have but one body, and this one has a mass m and at any assigned instant it has a velocity v in a given direction, we have, so long as no external force acts upon the body,

$$mv = \text{const.}$$

Since the mass is invariable, and the product of the mass and velocity is constant, it follows that the velocity must also be a constant quantity. Therefore the velocity of a body will remain constant so long as it is unacted upon by an external force. This is in agreement with what has been already shown in a different manner.

As a simple illustration again consider the case of the horse and cart which was discussed in Art. 5. When moving with an acceleration a, the effective force acting upon the system must equal the sum of the force of friction, F_f, and the force required to impart to the entire moving system the acceleration a. If the mass of the horse and cart be M, the force necessary to impart to it the acceleration a equals Ma. Thus, the effective force is $(F_f + Ma)$. Consequently, if the external force impressed on the cart, namely the force exerted by the horse, be represented by F_h,

$$F_h - (F_f + Ma) = 0.$$

If the motion is uniform, $a = 0$, and

$$F_h - F_f = 0.$$

PROBLEMS.

1.—A man weighing 140 lb. mounts a tram car weighing 2240 lb. moving at the rate of 8.5 ft. per sec. Find the change produced in the speed of the car. [0.5 ft. per sec.]

2.—An inelastic body of 900 kg. mass moving with a speed of 30 m. per sec. strikes an equal and similar mass moving 10 m. per sec. in the opposite direction. Find the velocity of the total mass after impact. [10 m. per sec.]

3.—A rifle bullet of mass 5 gm. is fired vertically upward into a block of wood of mass 1 kg. suspended from the ceiling. Due to the impact of the ball, the wooden block rises 2.5 cm. Find the velocity of the bullet. [14070 cm. per sec.]

4.—A bullet having a mass of 0.05 lb. passes horizontally through a block of wood of mass 4 lb. resting on a smooth level plane. The speed of the bullet is thereby changed from 1000 ft. per sec. to 750 ft. per sec. Find the speed of the wooden block after the impact. [3.125 ft. per sec.]

5.—A gun weighing 3000 kg. resting on smooth level tracks discharges a 30 kg. ball at an elevation of 30° with a muzzle velocity of 300 m. per sec. Find the velocity of the gun's recoil. [2.6 m. per sec]

6.—A shot of mass 20 lb. is fired from a gun of mass 2000 lb. and length 10 ft. The gun rests at the foot of an inclined plane rising 1 in 15. If the muzzle velocity of the shot be 1200 ft. per sec. in the horizontal direction, find the distance up the plane the gun will recoil, and also the average force exerted by the powder while the shot is in the gun. [36.7 ft.; 45000 lb. wt.]

7.—While running at 45 mi. per hr. the brakes are applied to an electric car in such a manner as to bring the car to rest in 500 ft. A man is standing in the middle of the track 75 ft. in front of the car when the brakes are applied. How close could he allow the car to come to him before he commenced walking at right angles to the track at the rate of 3 mi. per hr. to be just off the track by the time the car reached the point where he was standing? [71.7 ft.]

8.—Suppose the man in the preceding problem were 300 ft. away when the car commenced stopping, how close could he let it come before starting to move off the track? [49.4 ft.]

9.—Suppose the man in problem No. 7 were 400 ft. away when the car commenced stopping, how close could he let it come before starting to move off the track? [26 ft.]

10.—A man weighing 150 lb. is sitting in a car whose seats run lengthwise of the car. If the coefficient of friction between the man and seat is 0.56, will the man slide in the seat when the car starts with an acceleration of 18 ft. per sec. per sec.? If the car were moving with this acceleration and the man was just on the point of slipping, what would be the value of the coefficient of friction. [Yes; 0.5607.]

11.—A kite weighing 1 lb. is in equilibrium in the air when the plane of the kite is inclined 45° to the horizon and the wind blows in a horizontal direction against the kite with a force of 3 lb. wt. Find the tension and direction of the string. [1.57 lb. wt.; 18°26′ from the horizontal.]

12.—A body acted upon by a uniform force in 10 sec. describes a distance of 25 ft. Compare this force with the weight of the body. Also, find the speed acquired. [1:64; 5 ft. per sec.]

§ 4. *Circular Motion.*

85. Forces in Circular Motion.—Imagine a particle moving initially with constant velocity of magnitude s in the direction AB, (Fig. 63). to be deflected from its rectilinear path by the smooth circular rail BCD. Since the rail is smooth no force parallel to the rail is acting upon the particle; consequently the linear speed of the particle remains constant. It has been shown, (Art. 74), that at every point in the path of a body moving with uniform speed in the circumference of a circle, the velocity of the body has an acceleration directed toward the center of the path equal to the square of the linear speed divided by the radius of the path. Consequently, the force required to deflect the particle out of the rectilinear path into a circular path of radius r, is directed toward the center of its path and has a magnitude

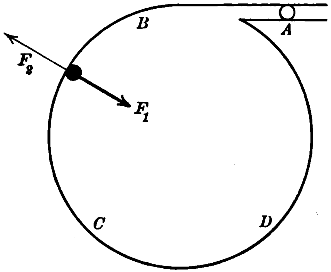

Fig. 63.

$$F_c = \frac{ms^2}{r}. \tag{59}$$

The force required to overcome the inertia of a body in deflecting it from a rectilinear path into a circular path is called *centripetal* (center seeking) force. In the above case, this force is due to the thrust of the rail against the particle.

The particle itself exerts on the agent which constrains it to move in a circular path a force $_rF_c$, which is equal in magnitude and opposite in direction to the centripetal force. This frequently called *centrifugal* (center flying) force is the reaction of the centripetal force and may be defined as the resistance which the inertia of a body in motion opposes to whatever deflects it from the rectilinear path. It should be kept in mind that the *centripetal* force acts upon the particle, while its reaction, the *centrifugal* force acts not upon the particle but upon the agent which constrains it to move in a circular path.

The force constraining a body to move in a circular path is balanced at every instant by the opposite tendency of the body to increase its distance from the center by pursuing the tangential path. If the body move with a greater speed, it will require a greater force to constrain it to move in a circular path of given radius. If the linear speed be increased while the centripetal force is kept constant, the radius of the circular orbit must increase. Always, the force which the inertia of a particle in motion opposes to whatever deflects it from its rectilinear path has the magnitude

$${}_rF_c = \frac{ms^2}{r} = 4\ m\ \pi^2\ r\ n^2 = \frac{4\ m\ \pi^2\ r}{T^2} = m\ w^2\ r, \qquad (60)$$

where m is the mass of the body, s is its linear speed, w is its angular speed, r is the radius of the circular path, n is the number of revolutions made per second, and T is the time occupied in making one revolution.

The centrifugal dryer used in laundries for drying clothes consists of a cylindrical drum, with perforated sides, capable of rapid rotation about a vertical axis. When rotated, the contents of the dryer tend to continue moving in a path tangential to the sides of the drum. The clothes cannot go through the perforations, but the drops of water can. In this manner most of the water is very quickly removed from the clothes. A similar machine is used to recover the oil mixed with the metal turnings and shavings of machine shops. In sugar refineries, sugar crystals are separated from admixed molasses by means of a similar machine.

From the value of the magnitude of the force developed when a body is rotated in a circular path (60), it is seen that if the rotated body consists of a mixture of particles of two different masses that are free to move amongst one another, the bodies of greater mass will be acted upon by a greater force than the bodies of small mass. Consequently, the two sets of particles will be separated, the heavier particles going farther from the center than the light particles. This is the principle employed in the centrifugal milk separator.

When going around a curve every particle of matter composing a bicycle and its rider tends to continue moving in a straight line and thereby to increase its distance from the center of curvature of the track. The friction between the road and tires prevents the lower portions of the wheels from retreating from the center of the curve, but the rider will fall outward if he does not incline his body toward the center of the curve. Again, when riding in a straight path, if he feels a tendency of the bicycle to fall over, he will right himself by turning the front wheel in such a direction that the center of the curved path thus formed is on the side toward which he felt himself falling.

It is due to this same principle that the circus performance called "looping the loop" illustrated in Fig. 64, is possible.

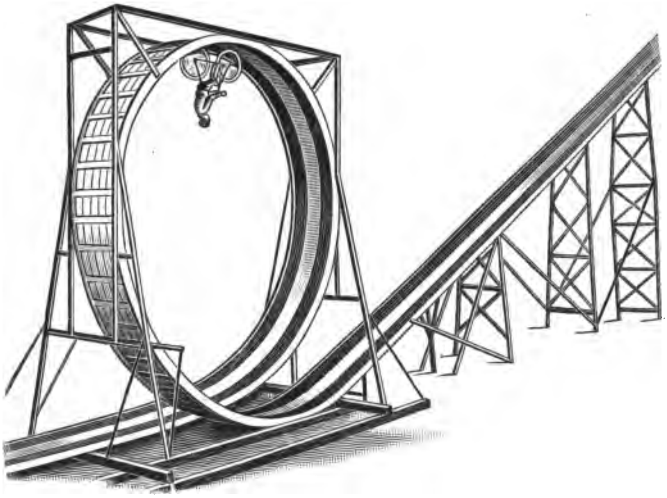

Fig. 64.

When a train is going around a curve, every particle composing its entire mass tends to continue moving in a rectilinear direction. The outer rail produces on the flanges of the wheels in contact with it a horizontal thrust directed toward the inside of the curve, and thus constrains the trucks of the cars to follow the curve. As there is nothing acting in a similar manner on the superstructure of the cars, they tend to increase their distance from the center of curvature of the track, thereby tending to lift the wheels off the inside rail. In practice the outside rail is somewhat higher than the inside rail, so that a component of the weight of the car acts toward the inside of the curve.

The elevation that the outer rail of a railway track on a curve of radius r must have in order that the flanges of the wheels on a car moving with a speed s shall produce no lateral thrust against the rails can be easily computed.

If the car is of mass m, it is acted upon by an effective force, *i. e.* one that imparts to it an acceleration, directed horizontally toward the center of the curve, and having a magnitude given by (59). It is also acted upon by its weight, mg, acting vertically downward, and the reactions, R_1 and R_2, normal to the plane of the roadbed. If the effective force be considered to be reversed in direction, then d'Alembert's principle (Art. 81) shows that the laws of equilibrium (Art. 43) can be applied. Thus, putting the algebraic sum of the horizontal components of the impressed and the reversed effective forces equal to zero, .

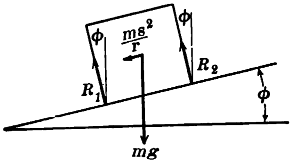

Fig. 65.

$$R_1 \sin \Phi + R_2 \sin \Phi - \frac{ms^2}{r} = 0.$$

Putting the sum of the vertical components equal to zero,

$$R_1 \cos \Phi + R_2 \cos \Phi - mg = 0.$$

Eliminating $R_1 + R_2$ from the two equations,

$$\Phi = \tan^{-1} \frac{s^2}{rg}.$$

It should be carefully noted that the effective force is a real force acting upon the car. In a previous paragraph it has been called the centripetal force. The "reversed effective force," however, is purely fictitious. It is taken to be of the same magnitude, in the reversed direction, and applied to the same body as the actual force. Consequently it is not the reaction of the centripetal force and cannot properly be called the *centrifugal* force.

PROBLEMS.

1.—Find the horizontal thrust on the rails when a 20 ton engine runs at 50 mi. per hr. on a curve of 1 mi. radius. [456.7 lb. wt.]

2.—A cord can just support a weight of 2 kg. What is the greatest length of it that can be used to whirl a mass of 500 gm. in a horizontal circle at a rate of 2 revolutions per sec.- [25 cm.]

3.—A fly-wheel is unbalanced by an amount equivalent to a mass of 20 lb. added at a point 3 ft. from the axis. If the flywheel revolves at the rate of 30 revolutions per min. find the magnitude of the force due to the lack of balance tending to lift the flywheel from its bearings. [18.4 lb. wt.]

4.—A pail of water is rotated in a vertical plane in a circle of 1 m. radius. Find the least period of revolution necessary to prevent spilling of the water. [2 sec.]

5.—A ring of mass 0.2 lb. is on a spoke of a wheel that revolves in a horizontal plane. If the coefficient of friction between the ring and the spoke is 0.1, and the distance of the ring from the axis of rotation is 18 in. find the number of revolutions per min. the wheel must make in order that the ring will just start to move toward the rim. [13.9.]

§ 5. *Systems of Units.*

86. Fundamental and Derived Units.—Since all physical phenomena are intimately connected, and the quantitative relations between them can be definitely formulated, it has been found possible to select a small number of elementary units from which all the other units required for measurement can be obtained. The units selected to form the basis for a system of units are called *fundamental* units. A unit whose magnitude is determined by the relations existing between the physical quantity to be measured and the fundamental units adopted, is called a *derived* unit. It is possible to select various sets of fundamental units. Thus, from the three units of length, mass, and time, all other desired units can be derived (1). Also, the units required for all physical measurement can be derived from the units of length, force, and time(1).

87. The Centimeter-Gram-Second Absolute System of Units.—A system in which certain units are arbitrarily selected as fundamental, and all the others are derived units connected with these by fixed physical relations, is called an *absolute system of units.* A system employing as fundamental units, units of length, mass, and of time, is called a *length-mass-time* system of units. If these quantities be the centimeter, gram, and second, the system is called the *centimeter-gram-second absolute system of units*, or more briefly, the *C. G. S. system.* For scientific work, this system is employed almost universally throughout the world. In the C. G. S. system the unit of force is a derived unit and is defined as that force which acting on a mass of one gram for one second will change its velocity by one centimeter per second in one second of time. This unit of force has a definite magnitude that is independent of the position of the body acted upon.

88. The Foot-Pound-Second Gravitational System of Units.—In engineering it is convenient to employ as fundamental units the units of length, force, and time. The unit of force usually adopted is the pound weight, or the kilogram weight. When the fundamental units are the foot, pound weight, and second, the system is called the *foot-pound-second gravitational system of units*, or more briefly, the *F. P. S. system of units.*

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(1) Although this statement is really subject to two exceptions, it is accurate so far as dynamics is concerned. In heat a unit of temperature is needed, and in electricity a unit of permeability.

Since the weight of a given body is not a fixed quantity at different places on the earth's surface, in order to make the gravitational system an absolute system of units the determination of the weight of the standard body must be made at a definitely assigned locality. The locality selected is sometimes London, sometimes Paris, and sometimes 45° N. latitude at sea level. Denoting the acceleration due to gravity at this place by the symbol g_o, the magnitude of the gravitational unit of force is g_o times the unit of force used in the corresponding length-mass-time absolute system.

In the gravitational system the unit of mass is a derived quantity. From (56),

$$F = ma,$$

it follows that the unit of mass in the F. P. S. system is that mass which acted upon by a force of one pound weight will be given an acceleration of one foot per second in one second. Since the unit of force in this system is g_o times the unit of force in the length-mass-time system, it follows that the unit of mass in this system is g_o times the unit of mass in the corresponding length-mass-time system of units.

In the C. G. S. length-mass-time system of units, the unit of mass is the *gram* and the unit of force is the *dyne.*

In the F. P. S. length-force-time system of units, the unit of mass is g_0 *pounds* and the unit force is the *pound weight.*

Two other systems of units are occasionally used. In the C. G. S. length-force-time system, the unit of mass is g_0 *grams*, and the unit of force is the *gram weight* ($= g_0$ dynes). In the F. P. S. length-mass-time system, the unit of mass is the *pound*, and the unit of force is the *poundal.*

CHAPTER VII.

CENTER OF MASS OR CENTER OF INERTIA.

89. The Center of Mass defined.—If a body or system of bodies be conceived to be divided into particles of equal mass, then that point whose distance from any given plane is equal to the average distance from that plane of all the constituent particles, is termed the *center of mass* or *center of inertia* of the body or system of bodies. In case the bodies composing the system have incommensurable masses, then they may be conceived to be divided into particles of as nearly equal mass as is desired by making the particles sufficiently small. This point has several peculiar and important properties, a few of which will here be considered.

90. *Location of the Center of Mass.—Consider a system composed of bodies of masses m_1, m_2, m_3, etc. Imagine each of these bodies to be broken up into a number of particles of equal mass m'. Suppose there are n_1 of these particles included in the mass m_1, n_2 in the mass m_2, etc. If the entire number of equal particles be N, then the mass of the system $M = Nm'$.

Also, let x_1 denote the mean distance, from any given plane, of the particles composing the mass m_1; let x_2 denote the mean distance, from the same plane, of the particles composing the mass m_2; and so on for the remaining parts of the system. Then if X represents the distance of the center of mass from the given plane,

$$
\begin{aligned}
X &= \frac{\text{lim.}\,[n_1\,x_1 + n_2\,x_2 + \text{etc.}]}{N} \\
&= \frac{\text{lim.}\,[m'\,(n_1\,x_1 + n_2\,x_2 + \text{etc.})]}{N\,m'} \\
&= \frac{\text{lim.}\,[m_1\,x_1 + m_2\,x_2 + \text{etc.}]}{M} \\
&= \frac{\text{lim.}\,\Sigma\,mx}{M}. \qquad (61)
\end{aligned}
$$

Similarly if Y and Z represent the distances of the center of mass of the system from any other two planes,

$$Y = \frac{\text{lim. } \Sigma\, my}{M}. \tag{62}$$

$$Z = \frac{\text{lim. } \Sigma\, mz}{M}. \tag{63}$$

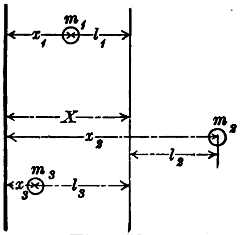

Fig. 66.

The three equations (61), (62) and (63) locate the center of mass of a body with reference to any three assigned planes of reference.

91. *Again, imagine a plane to pass through the center of mass parallel to one of the planes of reference. Let the distances of the several particles from the plane passing through the center of mass of the system be represented by l_1, l_2, l_3, etc.

Then

$$m_1\, x_1 + m_2\, x_2 + \text{etc.} = m_1\,(X + l_1) + m_2\,(X + l_2) + \text{etc.}$$

or, $\quad \text{lim. } \Sigma\, mx = X \text{ lim. } \Sigma\, m + \text{lim. } \Sigma\, ml.$

$$= X\,M + \text{lim. } \Sigma\, ml.$$

But, from (61), $\quad XM = \text{lim. } \Sigma\, mx.$

Therefore $\quad \text{lim. } \Sigma\, ml = 0. \quad (64)$

Consequently, *the algebraic sum of the products of the masses of the particles composing a body, and their corresponding distances from any plane passing through the center of mass of the body, equals zero.*

92. The Cenroid coincident with the Center of Mass.—If the motion of a rigid body be without angular acceleration, each particle at any instant has the same linear acceleration, and the resultants of the forces acting on the different particles are in the same direction. Let the masses of the very small particles of which the body may be considered to be composed be m_1, m_2, m_3, etc., and let their common linear acceleration be denoted by a. The parallel forces acting on the various particles are $F_1 = m_1\, a$, $F_2 = m_2\, a$, $F_3 = m_3\, a$, etc. Since these forces are in the same direction, the total resultant force, R, acting on the body is

$$R = \lim.[F_1+F_2+F_3+ \text{etc.}] = \lim.[m_1a+m_2a+m_3a+ \text{etc.}] = Ma$$

where M represents the total mass of the body.

It has been shown (Art. 25), that the moment of the resultant of a system of coplanar forces, about any axis perpendicular to their plane, equals the algebraic sum of the moments of the component forces about that axis. Proceeding in a somewhat similar manner it can be shown that the moment, about any axis, of the resultant of a system of forces not in one plane equals the algebraic sum of the moments of the component forces about that axis.

Through any selected point of reference, pass three planes that mutually intersect at right angles. Let the distances of the separate particles from one of these coordinate planes be x_1, x_2, x_3, etc. Let their distances from the other two planes be represented by y_1, y_2, y_3, etc., and z_1, z_2, z_3, etc., respectively. If X, Y, and Z represent the limit of the distances of the point of application of the resultant from the three coordinate planes, then,

$$\begin{aligned} RX &= \lim.\ [F_1\,x_1 + F_2\,x_2 + F_3\,x_3 + \text{etc.}] \\ &= \lim.\ [m_1\,a\,x_1 + m_2\,a\,x_2 + m_3\,a\,x_3 + \text{etc.}] \\ &= a \lim. \textstyle\sum mx. \end{aligned}$$

But since $R = Ma$, this equation becomes

$$MaX = a \lim. \textstyle\sum mx.$$

$$X = \frac{\lim. \sum mx}{M}. \tag{65}$$

Proceeding in the same manner, we obtain

$$Y = \frac{\lim. \sum my}{M}, \tag{66}$$

$$Z = \frac{\lim. \sum mz}{M}. \tag{67}$$

Equations (65), (66), and (67) locate the point of application of the resultant of a system of parallel forces, that is, the centroid of the parallel forces. A comparison of these equations with (61), (62), and (63) shows that *if the particles of a rigid body moving without angular acceleration are acted upon by forces whose resultants are all in the same direction, the centroid of this system of forces coincides with the center of mass of the body.*

Conversely, *if the resultant of all the forces acting on a rigid body is a single force acting at the center of mass, the motion of the body is without angular acceleration.*

93. The earth's gravitational attraction exerts a force on each particle of a body toward the center of the earth having a magnitude directly proportional to the mass of the particle. Consequently each particle tends to move toward the center of the earth with the same linear acceleration, *i. e.* without angular acceleration. Since the directions of the forces acting on the separate particles composing a body are practically parallel, it follows that the resultant force of gravity acting on a body passes approximately through its center of mass. In other words, *the weight of a body acts approximately at its center of mass.*

Hence if a body is suspended freely at any point, the line from the point of suspension to the center of the earth will pass approximately through its center of mass. This furnishes an experimental method for locating the center of mass of a rigid body. If the body is suspended in succession at two or more different points, the point of intersection of the lines of direction of the supporting string is approximately at the center of mass.

94. *Motion of the Center of Mass.—Consider any body composed of particles of masses m_1, m_2, m_3, etc. Let the distances of these component particles from any plane of reference be x_1, x_2, x_3, etc., and let the distance of their center of mass from the same plane be X. Then from (65),

$$X = \frac{\lim. \Sigma\, mx}{M}.$$

After a short time interval, τ, let the distances of the component particles from the plane of reference be x_1', x_2', x_3', etc.

The distance of the center of mass of the system will now be

$$X' = \frac{\text{lim.}\ \Sigma\ mx'}{M}.$$

If the first equation be subtracted from the second, and both sides of the result divided by τ, we obtain

$$\frac{X' - X}{\tau} = \frac{\text{lim.}\ \Sigma\ m\ (x' - x)}{M\ \tau}.$$

If the time interval τ be considered indefinitely short, the left side of this equation is the velocity, s_c, of the center of mass in a direction perpendicular to the given plane, and $\frac{x_1' - x_1}{\tau}$, $\frac{x_2' - x_2}{\tau}$, etc., are the velocities of the component particles in the same direction. Denoting the magnitudes of these latter velocities by s_1, s_2, s_3, etc., the above equation can be written

$$s_c = \frac{\text{lim.}\ \Sigma\ ms}{M}. \qquad (68)$$

Suppose that during the small time interval τ, the velocities of the various particles change to s_1', s_2', s_3', etc., respectively. Starting with (68) and using the same method by which this equation was derived, the value of the acceleration, normal to the plane of reference, of the center of mass of any material system, is shown to be

$$a_c = \frac{\text{lim.}\ \Sigma\ ma}{M}. \qquad (69)$$

The symbol lim. Σ *ma* means the limit of the sum of the components, in the direction of motion, of the effective forces acting on the particles of the system. This equation shows that, *for any material system, the effective force in any direction is the same as that for a particle of mass equal to that of the system, moving in that direction, with the acceleration of the center of mass.*

Let F_1, F_2, F_3, etc., be the components, in the direction normal to the plane of reference, of the external forces acting upon the system. Then, by D'Alembert's principle (Art. 81),

$$\text{lim. } \Sigma F - \text{lim. } \Sigma ma = 0.$$

Or, since from (69) lim. $\Sigma ma = Ma_c$,

$$\text{lim. } \Sigma F = Ma_c. \qquad (70)$$

Expressions like (70) can in the same may be obtained for motion away from two other coordinate planes, and from these three equations it follows that *the acceleration of the center of mass of any material system is the same as for a particle of mass equal to the mass of the system, all the external forces being transferred to this particle without change of magnitude or direction.*

This equation also shows that *the motion of the center of mass of any material system is not affected by the internal forces between the particles of the system, but only by external forces.*

If there are no external forces, that is, if lim. $\Sigma F = 0$, (70) shows that a_c also equals zero. That is, *no material system can of itself without the action of external forces change the motion of its center of mass.* This important theorem is called the Principle of the Conservation of the Center of Mass.

If a rapidly rotating fly-wheel should burst, each fragment would fly off in a line tangent to its path at the moment of the bursting, and the center of mass of the system of pieces would fall vertically, with uniform acceleration, in the line passing through the axis of the shaft.

A bomb-shell shot into the air traverses a certain path. If at any point of its path the bomb should explode, the fragments would so move that their center of mass would follow the same path, and in the same time, that it would have followed had the bomb not exploded.

It is impossible for an animal freely falling through the air to alter the position of its center of mass. The explanation of the ability of some animals to turn over while freely falling must be sought along some other line.

CHAPTER VIII.

THE MOTION OF A BODY UNDER THE ACTION OF A CONSTANT TORQUE.

§ 1. *Uniformly Accelerated Angular Motion.*

95. Angular Acceleration.—If a force act upon a stationary body capable of rotation about a fixed point not in the line of action of the force, the body will turn about the fixed point. If the body be already rotating, the application of the force will change the angular speed of the body about the fixed point. In either case a change in the angular speed of the body will be produced. When the angular speed of a rigid body about a fixed axis varies, the rate of change of the angular speed is called the *angular acceleration* of the body's motion. Thus, if during the time t, the angular speed about a given axis varies uniformly from w_o to w_t, there is a uniform acceleration during the interval equal to

$$\mathbf{a} = \frac{w_t - w_o}{t}. \tag{71}$$

The unit of angular acceleration is the change of angular speed at the rate of one radian per second in one second.

Howsoever the rotation of a body about a fixed axis may change during a given time, there must be a certain equivalent angular speed with which a uniformly rotating body would rotate through the same angle in the same time. If the rotation is uniformly accelerated, the equivalent uniform angular speed may, by the method of Art. 73, be shown to be the arithmetical mean of the instantaneous speeds at the beginning and end of the time considered. Consequently if the angular speed of the body changes uniformly during the time t from w_o to w_t, then the angle Φ swept through during this time by any line in the body and normal to the axis of rotation is

$$\Phi = \tfrac{1}{2}\,(w_o + w_t)\,t. \tag{72}$$

By means of (71) and (72), or of other equations obtained by combining them, can be solved any problem of uniformly accelerated angular motion involving the quantities Φ, w_o, w_t, t, and $\mathbf{a}$.

96. Angular accelerations can be represented by right lines in the same manner as angular velocities (Art. 67). Angular accelerations can be compounded and resolved by the parallelogram law precisely as can angular velocities (Art. 68).

97. The Relation between Angular and Linear Acceleration.—From (71), (42), and (43), we have

$$\mathbf{a} = \frac{w_t - w_0}{t} = \frac{\dfrac{s_t}{r} - \dfrac{s_0}{r}}{t} = \frac{r_t - s_0}{t\,r} = \frac{a}{r}. \tag{73}$$

Whence, the angular acceleration of any body is equal to the linear acceleration of any point of the body divided by the distance from that point to the axis of rotation.

98. Torque.—Whatever is capable of producing an angular acceleration is termed a *torque*. The quantity by which torque is measured will now be determined.

Imagine a massless rod AB (Fig. 67) capable of rotation about an axis through A normal to the plane of the paper. Let

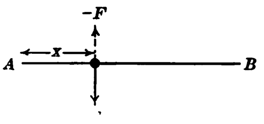

Fig. 67.

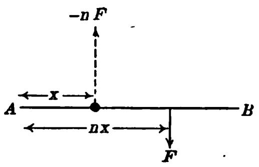

Fig. 68.

a particle of mass m be attached to the rod at a point distant x from the axis of rotation. If a force F be applied to m in such a way as to be always normal to both the rod and the axis of rotation, then the mass m will move about A in a circle of radius x with uniformly increasing speed, that is, with a uniform linear acceleration. If this linear acceleration be denoted by a, every line passing through the particle will have an angular acceleration, (73)

$$\mathbf{a} = \frac{a}{x}.$$

If F were to be made n times as great, a, and therefore **a**, would also be n times as great. Whence, if the distance of the particle from the axis of rotation is constant,

$$\mathbf{a} \propto F. \tag{74}$$

Now suppose that, in addition to F, there is applied to m an equal and opposite force $-F$. Then the acceleration with which m moves will be zero. If, however, the force F were applied at a distance nx from A (Fig. 68), it would require a force $-nF$ applied at m to balance it (Art. 24).

But if, under the action of F at a distance nx from A and $-nF$ at a distance x from A, the acceleration of m is zero, it follows that the acceleration which either of these forces alone would produce is equal and opposite to that which the other would produce. Consequently, F, acting at the distance nx from A produces the same acceleration of m that nF produces when acting at a distance x from A, that is, an acceleration n times as great as that produced by F acting at a distance x from A. Whence, if the force is constant,

$$\mathbf{a} \propto x. \tag{75}$$

Combining (74) and (75),

$$\mathbf{a} \propto Fx. \tag{76}$$

Therefore, *the magnitude of the angular acceleration of a body's motion produced by a force is directly proportional to the moment of the force about the fixed axis.*

Whence, torque is measured by the moment of the resultant force applied to the body.

§ 2. *Moment of Inertia.*

99. Angular Acceleration produced by a Uniform Torque. —The apparatus depicted in Fig. 69 consists of an easily turning spindle to the upper end of which various bodies can be attached. The lower end is provided with a drum *E*, to which a constant torque can be applied by means of a weight acting through a flexible cord arranged as shown in the engraving. The angular acceleration of the rotating system is proportional to the linear acceleration of the vertically falling mass; and this, in turn, is proportional to the distance the mass falls in a given interval of time.

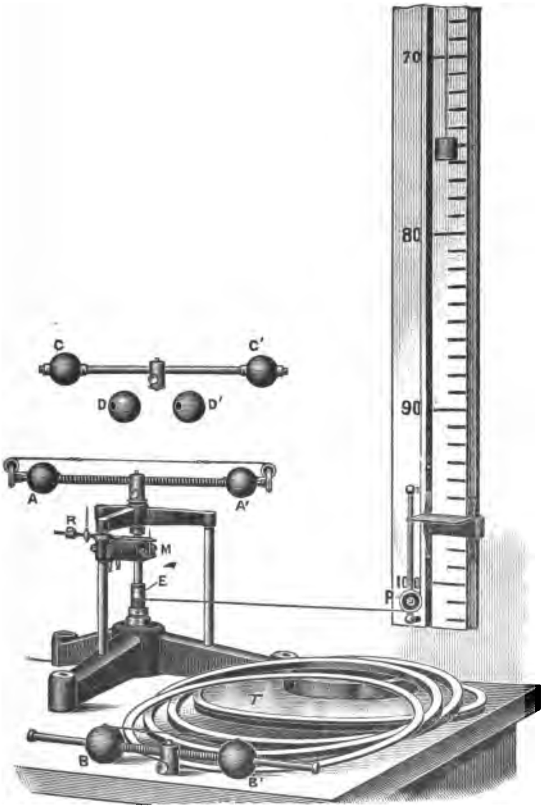

Fig. 69.

The body *AA'* consists of two similar spheres, connected by a spiral spring, and capable of sliding along a horizontal rod. These spheres can be drawn apart by means of a cord stretched over two small pulleys at the ends of the rod. If the spindle be released it will rotate with constant angular acceleration under the influence of the constant torque produced by the falling mass. If now, the cord holding the spheres apart be burned asunder, the stretched spring will pull the spheres toward the axis of rotation, and the angular acceleration of the rotating system will be greatly increased. The body BB' is arranged to take the place of AA' and consists of two similar spheres held apart by a spiral spring. These spheres can be drawn together by means of a cord. If this cord be burned asunder while the system is rotating, the spheres will fly apart, and the angular acceleration of the rotating system will be greatly diminished. These two experiments show that *the angular acceleration of a given body, under the action of a constant torque depends upon the distribution of the mass with respect to the axis of rotation.*

The body CC' consists of a horizontal rod with two solid spheres attached to the ends. These solid spheres can be removed and two hollow spheres, DD', of the same diameter, substituted for them. When this body is attached to the rotating spindle, and the solid spheres are in place, the angular acceleration of the moving system is found to be much less than when the hollow spheres are substituted for them. This shows that *the angular acceleration of a body of definite shape, under the action of a constant torque, depends upon the mass of the body.*

The relation between the magnitude and distribution of the mass of a body, the torque applied, and the angular acceleration developed, will now be considered. From the definition of inertia, the linear acceleration of a body's motion produced by the application of a force is always inversely proportional to the inertia of the body. It has been shown (Art. 78), that if a particle of mass m be acted upon by a force F, there will be produced a linear acceleration

$$a = \frac{F}{m}.$$

It has also been shown, (Art. 98), that if a torque be applied to a body, there will be produced an angular acceleration of its motion proportional to the torque applied. That is,

$$\mathbf{a} = \frac{F\,x}{K} = \frac{L}{K}. \tag{77}$$

where L is the torque, *i. e.*, the moment of the applied force about the axis of rotation, and K is some function of the inertia of the particle yet to be determined. From the experiment considered at the beginning of this article, it appears that this quantity K depends both upon the mass of the particle and upon its distance from the axis of rotation. Since the numerical measure of the importance of any physical agency on the rotation of a body is frequently called the *moment* of the particular agency, it is customary to call the quantity K, the *moment of inertia* of the particle. The value of this quantity, K, will now be determined.

Consider a particle of mass m, (Fig. 70), atached to a massless rod of length r capable of rotation about C. If a force F be applied at B in such a way as to be always normal to the rod and to the axis of rotation, the particle will move in the circumference of a circle with constant linear acceleration. In Art. 98 we have seen that the same acceleration would be produced if, instead of the force F applied at B, another force F' were applied to the particle of such a magnitude that its moment

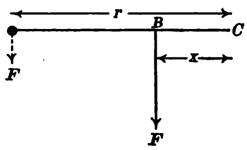

Fig. 70.

$$F'r = Fx.$$

Denoting the linear and angular accelerations of the particle by a and $\mathbf{a}$, respectively, we have, from (73),

$$F' = ma = m\mathbf{a}r.$$

Substituting this value of F' in the preceding equation, we obtain

$$\mathbf{a} = \frac{F\,x}{mr^2}.$$

Comparing this with (77), we obtain

$$\mathbf{a} = \frac{L}{K} = \frac{Fx}{mr^2}. \tag{78}$$

That is, the moment of inertia of a particle equals the product of the mass of the particle and the square of its distance from the axis of rotation.

It can be shown, although the proof will not be here given, that the moment of inertia of any body equals the limit of the sum of the moments of inertia of the separate component particles about the given axis when the number of particles is increased indefinitely. That is,

$$K = \lim. \Sigma (mr^2). \qquad (79)$$

For any given mass revolving about an axis at any fixed distance, the moment of inertia is a constant quantity quite independent of both the speed of rotation and the force acting. Consequently, *the angular acceleration of the motion of any body is numerically equal to the ratio of the moment of the force applied, to the moment of inertia of the body about the axis of rotation.*

100. Inertia and Moment of Inertia compared.—*Inertia* is the tendency of a body to keep its linear velocity of constant magnitude in an invariable direction. *Moment of inertia* is the tendency of a body to keep its angular velocity of constant magnitude about an invariable axis of rotation.

The measure of the *inertia* of a body is the sum of the masses of its component particles. The measure of the *moment of inertia* of a body about a given axis is the sum of the products of the masses of the particles composing the body and the squares of their respective distances from the axis of rotation.

When a *force* is applied to a body, there is produced a linear acceleration numerically equal to the ratio of the applied force, to the *inertia* of the body. When a *torque* is applied to a body, there is produced an angular acceleration numerically equal to the ratio of the moment of the applied force, to the *moment of inertia* of the body.

In short, moment of inertia plays the same role in rotation that inertia plays in translation.

101. Illustration of Moment of Inertia. The Falling Cat.—It is due to the application of (86) that a freely falling cat almost always alights on its feet.

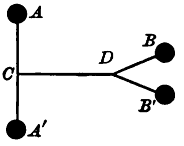

Fig. 71.

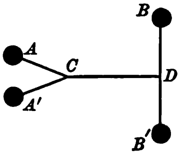

Fig. 72.

Imagine a body consisting (Figs. 71 and 72) of a rod *CD*, to the ends of which are hinged four rods carrying the masses *A*, *A'*, *B*, and *B'*. Assume that by means of some internal mechanism, a torque can be given to the body about the axis *CD*.

Equation (78) shows that for a constant torque, the angular acceleration produced is inversely proportional to the moment of inertia of the body; and (79) shows that the moment of inertia, about any assigned axis, of a body of given mass, can be considerably altered by small changes in the distances of its parts from the axis of rotation.

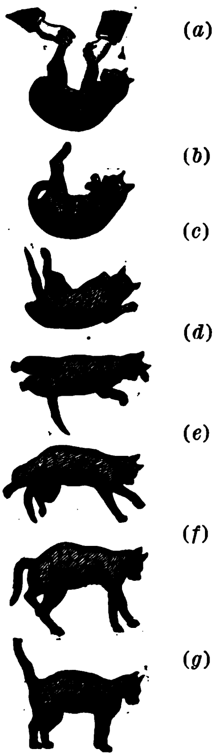

Fig. 73.

With the parts arranged as in Fig. 71, the moments of inertia of the left end of the body, about the axis *CD*, is greater than that of the rignt end of the body about the same axis. With the parts arranged as in Fig. 72, the reverse is true. Consequently, if, with the configuration of Fig. 71, a torque be developed about the axis *CD*, the left end of the body will experience a smaller angular acceleration than the right end: that is, in a given time the left end of the body will rotate through a smaller angle than the right end. With the parts arranged as in Fig. 72, the reverse is true. On this principle depends the ability of the cat to rotate his body while falling freely through the air.

Fig. 73 is an engraving made from a series of kinetoscope photographs of a freely falling cat. An inspection of these figures shows that the first operation was simultaneously to extend the hind legs perpendicular to the median axis of the body, and draw the fore legs close in to the body. A torque now applied about the median axis has resulted (Fig. 73*c*) in a rotation of the fore quarters nearly 90° in advance of the hind quarters. By drawing in the hind legs, extending the fore legs, and exerting another torque in the direction opposite to the previous one, the hind quarters have been rotated (Fig. 73*e*) in advance of the fore quarters. By this series of operations, an agile animal can produce a sufficient rotation of the entire body to enable it always to alight on its feet, even when falling from a comparatively small height.

102. The moments of inertia of bodies having regular geometrical shapes can be ecomputed.But since, in general, their computation involves mathematical methods that it would be inappropriate to use in a course having the limitations of the present one, such calculations will be omitted. For use in solving problems, values of the moments of inertia of two regular bodies about different axes are given below.

The moment of inertia of a uniform right solid cylinder of mass M and radius r, about its geometrical axis is

$$K_c = \tfrac{1}{2} Mr^2; \tag{80}$$

while about an element of its surface parallel to its geometrical axis, its moment of inertia is

$$K'_c = 1\tfrac{1}{2} Mr^2. \tag{81}$$

The moment of inertia of a cylindrical ring of mass M, external radius r_1 and internal radius r_2, about its geometrical axis is

$$K_r = \tfrac{1}{2} M \ (r_1^2 + r^2{}_2); \tag{82}$$

while about a line parallel to the geometrical axis distant r_3 from it, its moment of inertia is

$$R'_r = \tfrac{1}{2} M \ (r_1^2 + r_2^2) + M \ r_3^2. \tag{83}$$

The *radius of gyration* is the distance from the axis of rotation to the point at which, if all the mass of the body were concentrated, the moment of inertia of the body would be the same as when in its original configuration. For a cylindrical ring so thin that r_1 and r_2 are nearly equal, the radius of gyration with reference to its geometrical axis is approximately equal to $\tfrac{1}{2} (r_1 + r_2)$.

The comparison of moments of inertia is readily effected by means of the apparatus shown in Fig. 69. For this purpose the apparatus is provided with a light table T which can be attached to the vertical rotating spindle, and an electromagnetic release RM. By introducing the electromagnetic release, a battery, and a switch in a clock circuit, the circular table can be released at any clock beat, and thereafter the armature M will give a sharp click at each succeeding second.

The mass falling in front of the vertical scale has a uniform linear acceleration a, and the rotating system has a uniform angular acceleration **a**. If the falling body starts from rest, then in time t, it will fall through some distance x, such that $x = \tfrac{1}{2} \ a \ t^2$. [Obtained by combining (43) (44).]

If the spindle on which the cord is wound has a radius r, and the cord neither slips nor stretches, the angular acceleration of the rotating system is

$$\mathsf{a} = \frac{a}{r} = \frac{2\,x}{rt^2}\,.$$

Whence, the angular acceleration of the rotating body is directly proportional to the distance x passed over by the falling mass during any assigned time t.

If the torque produced by the weight of the falling mass acting on the spindle of radius r be denoted by L, and the moment of inertia of the rotating system be denoted by K, then, by (78), the angular acceleration is

$$\mathsf{a} = \frac{L}{K}.$$

Whence, combining these two equations,

$$x = \frac{L\,rt^2}{2\,K}.$$

Consequently, if the rotating body be acted upon by a constant torque for a given time, the falling body will pass through a distance inversely proportional to the moment of inertia of the rotating body.

Thus, by noting the distance traveled by the falling body in a given time when the rotating table is unloaded, and then when loaded by any two bodies, one after the other, the ratio of the moments of inertia of the two bodies is obtained.

PROBLEMS.

1.—A fly-wheel 10 ft. in diameter and weighing 5 tons is making 40 revolutions per min. when thrown out of gear. If the diameter of the axle is 6 in., and the coefficient of friction is 0.05, find the time required for it to come to rest. [4 min. 20 sec.]

2.—How long must a couple of moment 30 units act on a wheel whose moment of inertia is 25 units in order to make it rotate twice in a sec.? [10.47 sec.]

3.—To a fly-wheel weighing 400 lb. whose radius of gyration is 2 ft. a constant torque of 400 lb. wt.-ft. units is applied. If the fly-wheel start from rest, calculate the angular speed at the end of 10 sec. [80 radians per sec.]

4.—A cylinder rolls down an inclined plane of height h. Find its linear speed at the bottom. [$\sqrt{(4gh \div 3)}$.]

5.—A uniform cylinder 4 ft. in diameter and weighing 190 lb. rolls down a plane inclined 30° to the horizontal. Find the angular speed after it has rolled over a distance of 30 ft. [12.7 radians per sec.]

6.—A thin hoop starting from rest rolls down a 2 per cent. grade without slipping. Calculate how long it will take to go 400 ft., also find the velocity acquired in that time. [49.9 sec.; 16 ft. per sec.]

7.—A uniform cylinder of radius 1 ft. rolls down an inclined plane without slipping. At the foot of the plane the angular speed is 5 radians per sec. Find the height of the plane. [0.019 ft.]

8.—A uniform cylinder 1 ft. in diameter and weighing 25 lb. is rotating about its geometrical axis at the rate of 200 revolutions per min. Find the value of the constant force applied tangentially at the circumference which will stop it in 30 sec. [46.5 lb. wt.]

9.—A cord carrying a body weighing 100 lb. is wrapped about a 500 lb. cylindrical drum 2 ft. in diameter having a radius of gyration of 9 in. The heavy body being allowed to fall from rest through 30 ft., calculate the angular speed given to the drum. The friction of the bearings is equal to a force of 20 lb. wt. applied at a distance of 1 in. from the axis of rotation. [29.8 radians per sec.]

10.—A locomotive and tender of mass 90000 kg. runs around a curve of 450 m. radius at the rate of 60 km. per hr. If the track be horizontal, find the horizontal thrust on the rails. If a plummet be suspended from the ceiling of the cab of the locomotive, find its angle of deflection from the vertical. [34000 kg. wt.; 30°59′.]

11.—A train of mass 500 tons is moving around a curve of radius 500 ft. with a speed of 30 mi. per hr. Find the resultant thrust on the track, and also find the angle of elevation of the track that will prevent any tendency of the train to leave the track or wrench the rails. [61.1 tons wt.; 7°.]

12.—A bicyclist goes around a half-mile circular track with a speed of 0.35 mi. per min. What is his inclination to the vertical? [35°10′.]

13.—Find what period of rotation of the earth would cause bodies at the equator to have zero weight. The equational radius of the earth is 6370 km. [5048 sec.]

✓14.—A body is projected with a velocity of 20 ft. per sec. down a plane inclined at 25° with the horizontal and having a coefficient of friction of 0.4. Find the distance traversed in 2 sec. [3.8 ft.]

✓15.—Some cars start from rest down an incline of 1% and after passing over 1 mi. of track run out on level rails. Calculate how far they will travel on the level track; also the greatest speed acquired. Assume the frictional resistances to be 10 lb. wt. per ton. [41.2 ft. per sec.; 5345 ft.]

✓16.—A force equal to the weight of 1 kg. acting on a body continuously for 10 sec. causes it to start from rest and travel 10 m. Find the mass of the body. [49 kg.]

17.—Assuming that the volume of the alloy equals the volume of its constituents, find the density of an alloy made by mixing 70 parts by weight of a metal of density 8.9 with 30 parts of a metal of density 7.1. [8.27.]

CHAPTER IX.

ENERGY.

103. Definitions.—The accomplishment of motion against a resisting force is called *work*. It has been shown (Art. 19) that work is measured by the product of the force acting on the body moved and the resolved part of the displacement of the body in the line of action of the force. The rate of doing work, that is, the amount of work performed per second is called *power*. *Energy* has been defined (Art. 17) as stored work, or as the ability to do work. The quantity of energy possessed by a system of bodies, is the amount of work it can do against external forces in passing from its present condition to some standard condition.

The energy of a system may be due to the motion of part of it with reference to other parts. This type is called *kinetic energy*. Or the energy of the system may be due to stresses between different parts of the system. This type is called *potential energy*.

In certain cases work may be done by a body, not on account of the motion or the strained condition of the body, but on account of the heat energy stored in the body. For example, a body can do work either by expanding and thereby overcoming pressure, or by expanding and thereby liberating heat, (1) which in turn is converted into work. In the former case the work is due to a diminution of the potential energy of the strained body, while in the latter case, the work is due to a diminution of the *internal* energy of the body. Again, the heat absorbed when water is converted into steam is stored up as internal energy until the steam resumes the liquid form. Steam equals water plus a quantity of internal energy. According to the kinetic theory of matter, which will be considered when the subject of Heat is taken up, the internal energy of a body is a form of potential energy due to the arrangement of the ultimate parts of which a substance is considered to be composed.

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(1) Many supersaturated solutions expand on crystallizing with an evolution of heat.

Work and energy are measured in the same units. In the C. G. S. absolute system of units, the unit of work is the amount of work done when one dyne of force moves the body to which it is applied, in the direction of the force, through a distance of one centimeter. This unit is called the *dyne centimeter* or *erg*. 10^7 ergs is called a *joule*. In this system, the unit of power is the *erg per second*. 10^7 ergs per second is called a *watt*.

In the F. P. S. gravitational system, the unit of work is the amount of work done when a force of one pound weight moves a body in the direction of the force through a distance of one foot. This unit is called the *foot pound*. In this system the unit of power is the *foot pound per second*. 550 foot pounds per second is called a *horse power*.

104. Work done, and Power developed, by Forces and Torques.—If, in opposition to a uniform force F a body is moved a distance x along the line of action of the force, the work done is, (1),

$$W = Fx. \tag{84}$$

If the linear displacement x is effected in time t, with a constant linear velocity s, then the power developed by the force is

$$P \left[= \frac{W}{t}\right] = \frac{Fx}{t} = Fs. \tag{85}$$

Consider a body capable of rotation about a fixed axis passing through the point C to be acted upon by a force F having a constant moment Fl about the axis of rotation. When the body has turned through an angle Φ radians, the point of application of the force will have moved a distance $l\Phi$ along the line of action of the force. Consequently, the work done by the torque is

$$W = Fl\,\Phi = L\,\Phi. \tag{86}$$

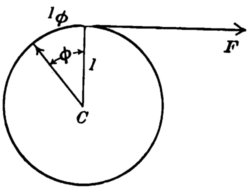

Fig. 74.

If the angular displacement Φ is effected in time t, with a constant angular velocity w, then the power developed by the torque is

$$P\left[=\frac{W}{t}\right]=\frac{L\,\Phi}{t}=L\,w. \tag{87}$$

Problem.—While a cut is being made on a 2-inch iron shaft in a lathe making 20 revolutions per minute, the power expended at the cutting tool is found to be 0.15 H. P. Find the force acting on the cutting tool.

Solution.—

$$\text{H. P.}=\frac{\textit{ft. lb. per sec.}}{550}.$$

The work done per second equals the product of the force acting on the cutting tool and the distance traveled in one second by a point in the circumference of the shaft.

$$\text{Distance traveled}=\frac{\pi}{6}\ \text{ft. per revolution.}$$

$$=\frac{20\,\pi\ \text{ft.}}{6\times 60\ \text{sec.}}=0.174\ \text{ft. per sec.}$$

Whence

$$0.15=\frac{0.174\,F}{550}.\quad \text{Therefore } F=474\ \text{pounds' weight.}$$

105. Kinetic Energy.—Imagine that in t seconds a body of mass m is brought from rest up to a speed s by the application of a uniform force F. If, during this time the body moved through a distance x with a uniform acceleration a, the work done upon the body is

$$W\ [=Fx]=max.$$

Combining (43) and (44), and remembering that the body starts from rest, we find

$$ax=\tfrac{1}{2}\,s^2.$$

Whence

$$W=\tfrac{1}{2}\,ms^2. \tag{88}$$

Due to its motion, the body now possesses an amount of energy which can be determined by measuring the amount of work it can perform. Suppose that in overcoming a constant force $-F'$, the body is brought to rest in t' seconds; and that during this time the body traversed a distance x' with a uniform acceleration $-a'$. It follows that when moving with the speed s, the kinetic energy of the body was

$$W' \; [= -F' x'] = -m a' x'.$$

Combining (43) and (44), and remembering that the final speed of the body is zero, we find

$$-a'x' = \tfrac{1}{2} s^2.$$

Whence

$$W' = \tfrac{1}{2} ms^2. \tag{89}$$

Consequently

$$W = W' = \tfrac{1}{2} ms^2 \text{ units of work.} \tag{90}$$

This equation shows that *the kinetic energy of a body is equal to the work done upon it in bringing it from rest to its present speed, and that this equals one half the product of the mass of the body and the square of its linear speed.*

106. In the preceding article an expression was obtained for the kinetic energy of a body moving with a motion of pure translation. The kinetic energy of a rigid body rotating about a fixed axis with angular speed w will now be considered.

At any given instant any particle of mass m at a distance r from the axis of rotaton is moving with a certain linear speed s. The kinetic energy of this particle is $\frac{1}{2}\, ms^2$ units of work. But since $s = wr$, (42), the kinetic energy of the given particle equals

$$\tfrac{1}{2}mw^2r^2.$$

Now the energy of the whole body equals the sum of the energies of its constituent particles. Consequently the kinetic energy of the rotating body equals

$$W_{ro} = \tfrac{1}{2}\, w^2 \text{ lim. } \Sigma\, mr^2 = \tfrac{1}{2}\, w^2 K \text{ units of work.} \tag{91}$$

In case a rigid body has a motion both of translation and of rotation, its kinetic energy consists of two parts,—one given by (90) and another given by (91).

107. Potential Energy.—Consider the particular case where the potential energy of a body is due to its position above the surface of the earth. Imagine a body of mass m to start from C and fall to a point A. Let the initial position of the body be at a height h above the horizontal plane passing through its final position. The work done by the body in moving from C to A along the path CA, (Fig. 75), is

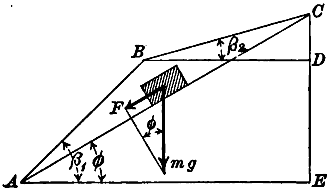

Fig. 75.

$$mg \sin\Phi(AC) = mg(CE) = mgh.$$

Similarly, the work done by the body in moving from C to A along the path CBA is

$$mg \sin \beta_1(AB)+mg \sin\beta_2(BC)=mg(ED)+mg(DC)=mgh. \quad (92)$$

Consequently, *the work done by a body moving from one position to another is independent of the path traversed.*

The potential energy of a body is measured by the work it can do in going from its present position to some standard position. In the above example, the difference between the potential energy of the body when at C and when at A equals mgh. That is, if A is the standard position from which potential energy is reckoned, the potential energy of the body when at C is mgh. Taking the earth's surface as the standard for reference, and neglecting the variation of the acceleration due to gravity at different positions on the earth's surface, the potential energy of a body due to gravitation is the same at all points at equal heights above the surface of the earth.

In strictness, it is not correct to speak of the potential energy of a *body* because the energy really belongs to whatever agent is under stress. But in order to avoid such circumlocutions as "the potential energy of the strained medium between the parts of the given system of bodies" custom has sanctioned such expressions as "the potential energy of the given body with respect to the earth." When the context shows what is taken to be the standard position from which potential energy is reckoned, it is common to speak simply of the "potential energy of the body."

108. The Potential Energy of a System tends to become a Minimum.—When a body moves in opposition to the resultant force upon it, positive work is done against the force, *i. e.*, the potential energy of the system of which the body forms a part is increased. Conversely, when a body moves in the same direction as the resultant force acting upon it, the potential energy of the system is diminished. Now when a motion takes place between the parts of a system due to the action of a force between them, the direction of the displacement is in the direction of the force. Consequently, *if under the action of a force between two parts of a system of bodies a motion is produced, the direction of the displacement will be such as to produce a diminution in the potential energy of the system.*

If work is required to separate two bodies, the bodies will, if left to themselves, tend to approach one another. The direction of the line of approach may not be coincident with the path along which they were originally displaced:—it will always be in that direction along which the change in the potential energy of the system, per unit distance traversed, is the greatest.

109. The Sum of the Kinetic and Potential Energies is Constant.—That greatest of all the generalizations of physics—the principle of the conservation of energy—has been enunciated by Maxwell in the following form:—"The total energy of any material system can neither be increased nor diminished by any action between the parts of the system, though it may be transferred into any of the forms of which energy is susceptible." Since energy can be only kinetic and potential, it follows as a direct deduction from the above principle, that, so long as no external forces act upon a system, the sum of its kinetic and potential energies must be a constant quantity. A simple example is a body falling freely under the influence of gravitation.

Consider a body of mass m which was initially at a height h above the ground. An expression will be found for its total energy at the moment it has fallen to a point at a height x above the ground. Let s be its speed at this instant. At the position considered, the total energy of the body, that is, the sum of its kinetic and potential energies is

$$\tfrac{1}{2}\, ms^2 + mgx.$$

Combining (43) and (44) by eliminating t, and remembering that the initial speed of the body was zero, the speed of the body after having fallen through the distance $(h - x)$ is given by the equation,

$$s^2 = 2\ g\ (h - x).$$

Therefore, a body falling from a height h above the earth has at any point of its path distant x from the earth a total energy equal to

$$\tfrac{1}{2}\ m\ 2\ g\ (h - x) + mgx = mgh = \text{a constant quantity.} \qquad (93)$$

When $x = h$, all of the energy is potential; when $x = 0$, all of the energy is kinetic; when x has any intermediate value, the energy is partly potential and partly kinetic. If all the energy possessed by a body at the bottom of its fall be utilized in raising it against gravity, the body will rise to its original height.

If one keeps clearly in mind that potential energy always implies a system under stress, and does not refer to a single body, one will not be confused by such a seeming paradox as the following: "when a body falls freely gravity does work upon it, thus continually increasing its energy, yet at every point of its fall the sum of its kinetic and potential energies is the same as when it started to fall.

PROBLEMS.

1.—Find the H. P. necessary to turn a 2 in. shaft 100 revolutions per min., if it exerts on the bearings a force of 1500 lb. wt. and the coefficient of friction is 0.01. [0.024 H. P.]

2.—Calculate the H. P. of a locomotive which is moving at the rate of 20 mi. per hr. up an incline which rises one foot in one hundred, the mass of the locomotive and load being 60 tons, and the frictional resistance 12 lb. wt. per ton. [108.8 H. P.]

3.—Find the H. P. required to haul a car weighing 100,000 lb. at the rate of 30 mi. per hr. up a 2 per cent. grade when the resistance of friction is 5 lb. wt. per 1000. [200 H. P.]

4.—A fire engine can project a stream of water from a nozzle 2 sq. in. in area to a height of 40 ft. If the work lost by friction equals the work expended in projecting the water, find the H. P. expended by the engine. [3.18 H. P.]

5.—A 500 lb. fly-wheel has a radius of gyration of 2½ ft. Calculate what H. P. applied for 1 min. will develop an angular speed of 2½ turns per sec. [0.365 H. P.]

6.—A freight car weighing 40000 lb. is moving along a level track at a speed of 30 mi. per hr. There are eight wheels on the car each weighing 562.5 lb. Find in foot-pounds, the total amount of energy stored in the car. [3,116,987 ft. lb.]

7.—By running along a level track and pushing uniformly against a hand-car for a distance of 50 ft. a man sets the car into motion and gives it a speed of 3 mi. per hr. If the hand-car with its load weighs 300 lb., and the frictional resistance is 3 lb. per 100, find the force exerted by the man. [10.7 lb. wt.]

8.—A ball weighing 5 oz. and moving with a speed of 1000 ft. per sec. pierces a shield and moves on with a speed of 400 ft. per sec. Find the energy lost by piercing the shield. [4076 ft. lbs.]

9.—How many pounds of coal can be elevated 50 ft. in 1 hr. by a 10 H. P. engine, if the friction of the hoisting mechanism absorbs an amount of energy equal to that required to raise the coal? [198000 lb.]

10.—A fire engine pump is provided with a nozzle the sectional area of which is 1 sq. in., and water is projected through the nozzle with an average speed of 130 ft. per sec. Find, (a), the number of cu. ft. discharged per sec., (b), the mass of water discharged per min., (c), the H. P. of the engine required to drive the pump, assuming the engine to have an efficiency of 70%. [0.9 cu. ft.; 1.51 tons; 31 H. P.]

11.—A 10 H. P. water turbine transforms 55% of the energy of a water fall into useful work. If the height of the fall be 100 ft., find the mass of water that falls in each sec. [100 lb. per sec.]

12.—The cylinder of a steam engine has a diameter of 15 in.; the stroke is 3 ft.; the number of strokes is 77 per min.; the mean pressure of the stream is 40 lb. per sq. in. What is the H. P. of the engine? [49.5 H. P.]

13.—A 60 ton locomotive runs one mile with constant speed. If the resistance be 8 lb. wt. per ton find the work done by the locomotive, (a), on a level track, (b), on a 1% grade. Also find the relation of the work done against gravity to the work done against friction on a 1% grade. [1267 ft. tons; 4435 ft. tons; 5 to 2 approx.]

14.—Find the H. P. required of a locomotive to haul a train of 100 tons at 30 mi. per hr., the resistances amounting to 8 lb. wt. per ton; (a), on a level road, (b) up a 1% grade, (c), up a 2% grade. [64 ; 224 ; 384.]

15.—The travel of the table of a planer which cuts both ways is 9 ft. If the resistance to be overcome while cutting is 400 lb. wt., and the number of double strokes is 80 per hr., find the H. P. absorbed while cutting. [0.22 H. P.]

16.—A cut of 0.06 in. depth is made on a 4 in. iron shaft making 10 revolutions per min.; the traverse feed is 0.3 in. per revolution; the pressure on the tool is 435 lb. wt. Find the H. P. expended at the tool. [0.138 H. P.]

17.—A man is riding a bicycle on a level road at a speed of 10 mi. per hr. If he comes to a 10% grade and takes his feet off the pedals how far will the bicycle go up the hill? [33.8 ft.]

18.—A man is riding a bicycle on a level road at a speed of 12 mi. per hr. On coming to a hill he finds that upon removing his feet from the pedals the wheel will go up the slope for a distance of 50 ft. Find the grade of the hill. [9.73%.]

CHAPTER X.

MOTION UNDER THE ACTION OF A VARIABLE FORCE.

§ 1. *Simple Harmonic Motion of Translation.*

110. Simple Harmonic Motion of Translation defined.— A very important type of motion is the one produced by a force acting on a body varying directly with the distance of the body from a certain position of equilibrium. Consider a small block resting on a smooth horizontal table and attached to two light similar horizontal springs, as shown in the figure. When at rest, the block will be at some point C. If the block be displaced in the direction of the axis of the springs to some point X', a force will act upon the block causing it to move toward its position of equilibrium. During its motion toward this point, the potential energy of the system due to the distortion of the spring diminishes until, when at the point C it becomes zero. At this point the entire energy is kinetic, and the body passes through its position of equilibrium and attains some position as at X. This action will then be repeated, the body vibrating back and forth from X to X'.

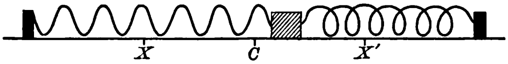

Fig. 76.

It is found by experiment that the force required to distort a spring or other elastic body, through a small range, is directly proportional to the displacement produced (Hooke's Law). And since at any instant the acceleration is directly proportional to the force acting, it follows that at any instant the linear acceleration of the motion of the small block is directly proportional to its distance from the point C. If displacements, accelerations, and forces are called positive when directed from C toward X', and negative when directed from C toward X, the value of the acceleration is

$$a = -cd, \tag{94}$$

where d is the linear displacement of the body from the equilibrium position, and c is a constant whose value is still to be determined.

The motion above described is that which occurs in the case of the prongs of a tuning fork, a plucked stretched string, the air in a sounding organ pipe. It is called simple harmonic motion of translation. *Simple harmonic motion of translation* is a reciprocating motion having at every instant an acceleration which is directed toward the center of its path and which varies directly with the distance of the moving body from that point.

Substituting the value of a, given above, in the fundamental force equation, $F = ma$, we obtain

$$F = -mcd. \qquad (95)$$

Consequently, simple harmonic motion can also be defined as the motion of a body which at every instant is urged toward a position of equilibrium with a force proportional to its linear displacement from that point.

The time which elapses between two consecutive passages of the oscillating body in the same direction through any given point of its path, is called the *period* of the simple harmonic motion. The maximum distance attained by the oscillating body from its position of equilibrium, is called the *amplitude* of the simple harmonic motion.

111. Relation between Uniform Circular Motion and Simple Harmonic Motion.—Let a particle P' move with uniform speed in the circumference of a circle $P'A'B'$, and let P be the projection of this point on any right line AB. As P' moves with uniform speed in the circumference of the circle, its projection P oscillates back and forth through a middle position C between two extreme positions A and B. The sort of motion described by P along the line AB will now be investigated.

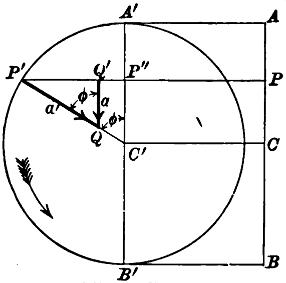

Fig. 77.

As the particle P' moves with unifrom speed s' in the circumference of a circle of radius D, there is directed toward the center of the circle an acceleration having a magnitude, (51),

$$a' = \frac{s'^2}{D}.$$

In Fig. 77, let this acceleration be denoted in magnitude and direction by the line $P'Q$. From the principle of the composition and resolution of accelerations (Art. 71), it follows that the acceleration of the point P is the component, in the direction AB, of the acceleration of the particle P'. In the diagram, the line $Q'Q$ represents the component in the direction AB of the acceleration $P'Q$.

The magnitude of $Q'Q$ is

$$a' \cos \Phi = \frac{s'^2}{D} \cos \Phi = \frac{s'^2}{D} \frac{(C'P'')}{(C'P')} = \frac{s'^2}{D^2} (CP). \qquad (96)$$

Denoting the displacement CP of the point P from the center of its path by d, and remembering that

$$s' = \frac{2 \pi D}{T}$$

where T represents the time occupied by P' in traversing the circumference of the circle, and counting all distances, velocities, and accelerations as positive when they are directed from C toward A, and as negative when directed from C toward B, we have the acceleration of P at any point of its path

$$a = -\frac{s'^2}{D^2} (d) = -\frac{4 \pi^2}{T^2} (d) \qquad (97)$$

$= -$(a constant quantity) (the displacement of P from the center of its path).

This fact, in connection with Art. 110, shows that, *if a point moves with uniform speed in the circumference of a circle, the projection of the point on any straight line moves with simple harmonic motion of translation.*

112. Velocity and Acceleration of a Particle moving with Simple Harmonic Motion of Translation.—The important fact derived in the preceding paragraph furnishes a simple means for the study of harmonic motion. From the principle of the composition and resolution of velocities (Arts. 60-63), it follows that the velocity of the point P, (Fig. 77), is the component in the direction AB of the velocity of the particle P'.

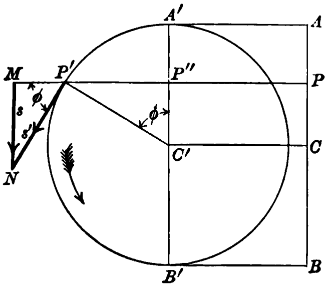

Fig. 78.

Denote the speed of the particle P' by s' and that of the point P by s. Let the velocity of the particle P' at any chosen instant be represented in direction and magnitude by the line $P'N$. Then the component of the velocity of P' in the direction AB is represented in direction and magnitude by the line MN. If T be the time occupied by P' in going once around the circumference of the circle of radius D, and if d be the displacement, CP, of P from the center of its path, then

$$s = s' \sin \Phi = \frac{2\pi D}{T} \sin \Phi = \frac{2\pi}{T}(P'P'') = \frac{2\pi}{T}\sqrt{(D^2 - d^2)} \qquad (98)$$

is the magnitude of the velocity, at any instant, of the point P moving with simple harmonic motion of translation.

Again, from (97), we have for the acceleration of a body moving with simple harmonic motion of translation

$$a = -\frac{4\pi^2 d}{T^2}. \qquad (99)$$

Or, solving for T, the period of a simple harmonic motion is

$$T = 2\pi\sqrt{\left(-\frac{d}{a}\right)}. \qquad (100)$$

113. Displacement of a Particle moving with S. H. M. of Translation.—From Fig. 78, the displacement of a particle at P is

$$d = D \cos \Phi = D \sin P'C'C.$$

If the time occupied in moving through the distance d be denoted by t, and the time of a complete vibration by T, then

$$\frac{P'C'C}{2\pi} = \frac{t}{T} \text{ or } \angle\, P'C'C = \frac{2\pi t}{T}.$$

Whence

$$d = D \sin \frac{2\pi t}{T}, \qquad (101)$$

where D is the amplitude of the vibration.

§ 2. *Simple Harmonic Motion of Rotation.*

114. Simple Harmonic Motion of Rotation defined.—If a body suspended by a vertical wire be rotated through a small angle about a line coincident with the axis of the wire, it is found by experiment that a force of restitution is developed in the wire having a moment L about the axis of rotation directly proportional to the angular displacement Φ. It follows that the angular acceleration of the body's motion at any instant is given by the relation

$$\mathbf{a} = -k\,\Phi, \tag{102}$$

where k is a constant whose value is still to be determined.

Since, from (77), $L = K\,\mathbf{a}$, where K is the moment of inertia of the rotating body,

$$L = -K\,k\,\Phi. \tag{103}$$

The angular motion of a body which at any instant is urged toward a position of equilibrium by a torque proportional to its angular displacement from that position is called *simple harmonic motion of rotation.*

Thus, if a body freely suspended by a vertical wire be rotated through a small angle about an axis coincident with the axis of the wire, and be then released, the body will vibrate with simple harmonic motion of rotation. Another example of the same motion is furnished by the balance wheel of a watch.

115. Angular Velocity and Angular Acceleration in Simple Harmonic Motion of Rotation.—It will be noticed that (102) and (103) are perfect analogues of (94) and (95). The magnitudes of the angular velocity and the angular acceleration of a body moving with simple harmonic motion of rotation can be obtained by a simple transformation of the latter equations.

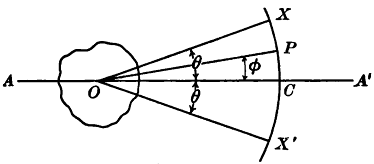

Fig. 79.

In Fig. 79, let AA' be the position of equilibrium of a certain line of the body which vibrates with simple harmonic motion of rotation, about an axis through O perpendicular to the plane of the diagram. While the line AA' vibrates about O with simple harmonic motion of rotation, every point in the line will oscillate about its position of equilibrium with simple harmonic motion of translation. Consider the linear motion of some point C of the line AA' distant r from the axis of vibration passing through O. Denote the amplitude of the linear motion of the selected point, CX $[= CX']$, by the symbol D; let its linear displacement from the position of equilibrium when at some point P be denoted by d.

Substituting in (99) the value of a obtained from (73),

$$\mathbf{a}\, r = -\frac{4\,\pi^2\, d}{T^2}.$$

Let Φ denote the angular displacement of the body when the given point is at the position P. From the figure,

$$\Phi \left[= \frac{CP}{OC}\right] = \frac{d}{r}.$$

Consequently, the angular acceleration of the body at the instant when the angular displacement of any given point from its position of equilibrium is Φ, is

$$\mathbf{a} = -\frac{4\,\pi^2\,\Phi}{T^2}. \tag{104}$$

Since, from (77), $L = K\,\mathbf{a}$, the period of a simple harmonic motion of rotation is

$$T = 2\,\pi\,\sqrt{\left(-\frac{\Phi}{\mathbf{a}}\right)} = 2\,\pi\sqrt{\left(-\frac{K\,\Phi}{L}\right)}. \tag{105}$$

The angular speed of the body at the instant when the angular displacement of any given point from its position of equilibrium is Φ, will now be determined. Let θ represent the amplitude of the angular displacement of the body. Let w represent the angular speed of the body at the instant when the angular displacement is Φ. On substituting in (98) values of s, D, and d obtained from the relations

$$w = \frac{s}{r}, \qquad \theta = \frac{CX}{OC} = \frac{D}{r}, \qquad \Phi = \frac{CP}{OC} = \frac{d}{r}.$$

we obtain for the angular speed of a body, moving with a simple harmonic motion of rotation of amplitude θ, at the instant when the angular displacement of the body from its position of equilibrium is Φ,

$$w = \frac{2\pi}{T} \sqrt{(\theta^2 - \Phi^2)}. \tag{106}$$

116. Displacement of a Point moving with S. H. M. of Rotation.—From Fig. 79,

$$\theta = \frac{D}{R} \quad \text{and} \quad \Phi = \frac{d}{R}.$$

Substituting in (101) the values of d and D obtained from these relations, the value of the angular displacement of the point P at any time t is

$$\Phi = \theta \sin \frac{2\pi t}{T}. \tag{107}$$

PROBLEMS.

1.—A tooth in the blade of a reaper describes a simple harmonic motion of 1.5 in. amplitude in a period of 0.2 sec. Find its speed when at a point 0.5 in. from the center of its path. [44.4 in. per sec.]

2.—A point moving with simple harmonic motion has a maximum speed of 10 ft. per sec. and amplitude of 6 ft. Find the time of a complete vibration. [3.77 sec.]

3.—The drive wheels of a locomotive whose piston has a stroke of 2 ft. make 185 revolutions per min. Assuming that the piston moves with simple harmonic motion, find the speed of the piston relative to the cylinder-head, when at the center of its stroke. [19.37 ft. per sec.]

4.—The scale-pan of a spring balance is vibrating vertically with a simple harmonic motion having an amplitude of 3 in. and a period of 2 sec. for a complete up-and-down swing. What will be the downward thrust on the scale-pan produced by a one pound mass when at a point one half of an inch from the top of a swing [0.936 lb. wt.]

5.—A scale pan attached to the lower end of a vertical spiral spring vibrates up and down with simple harmonic motion. If the period of one complete vibration is one sec. find the greatest amplitude of vibration that will permit objects resting on the scale pan to remain in contact with it throughout the entire vibration. [0.813 ft.]

6.—A block of iron of mass 100 lb. is caused to vibrate with simple harmonic motion by means of a spring. If the amplitude of vibration is 12 in. and the time of a complete vibration is 0.8 sec. find the maximum kinetic energy of the block. [3084 ft. lbs.]

§ 3. *The Pendulum.*

117. Period of a Simple Pendulum.—One day while attending service in the cathedral at Pisa, Galileo was impressed by the observation that the great swinging chandelier (Fig. 80) vibrated in apparently the same time whether the amplitude of vibration was large or small. In order to test the accuracy of this important observation, the next day Galileo returned to the cathedral determined to measure the period of oscillation of the chandelier, first just after it had been lighted, and again after the amplitude of swing had died down considerably. As there were no watches or other portable instruments for measuring time in those days, Galileo used his pulse-beats as the standard of time for determining the period of the swinging chandelier. The result of this measurement confirmed him in his opinion that the period of a swinging pendulum is independent of the amplitude of swing, so long as that amplitude is not excessive. The practicability of using a pendulum to mark off equal intervals of time now became evident and the application to clocks quickly followed. The reason that the pendulum oscillates with the same period whatever be the amplitude of its swing, so long as this amplitude is not excessive, will now be studied.

Fig. 80.

A *simple* or *mathematical* pendulum consists of a heavy particle suspended by a massless string. Consider the simple pendulum OB (Fig. 81), consisting of a body of mass m suspended by a massless string of length l. Let the linear displacement, AB, of the pendulum bob from its position of equilibrium be denoted by d. Two forces act on the bob B, viz. its weight mg, represented in the figure by BD, and the tension in the string. This tension, however, is normal to the path of the bob and so has no effect upon its motion. The force tending to restore the bob to its position of equilibrium, A, is a component of the weight mg, and is

$$F = -mg \sin \Phi.$$

When Φ is small,

$$\sin \Phi = \Phi = \frac{AB}{OA} = \frac{d}{l} \quad \text{(approx.)}$$

Whence,

$$F = -\frac{mgd}{l} \quad \text{(approx.)}$$

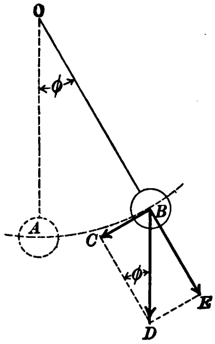

Fig. 81.

This equation shows that, when the amplitude of vibration is small, the pendulum bob is urged toward a position of equilibrium with a force proportional to its linear displacement from that point. Consequently such a pendulum moves with approximately simple harmonic motion.

Since $F = ma$, the instantaneous linear acceleration of a pendulum bob at any position of its path is

$$a = -\frac{gd}{l} \text{ (approx.)} \qquad (108)$$

Now in simple harmonic motion, the period of vibration is, (100),

$$T = 2\pi\sqrt{\left(-\frac{d}{a}\right)}.$$

Consequently, if during the time T a pendulum makes n complete vibrations, the period of a simple pendulum is

$$T = \frac{1}{n} = 2\pi\sqrt{\frac{l}{g}} \text{ (approx.)} \qquad (109)$$

From this equation it is seen that when the amplitude of vibration of a simple pendulum is so small that $\sin \Phi$ is approximately equal to Φ, the period of vibration of the pendulum is independent of the amplitude. This equation is the one ordinarily used in determining the value of the acceleration due to gravity. The closeness of the approximation in (109) is shown by the fact that if the length of the pendulum is about 100 cm., and the amplitude of vibration is about 3 cm., the error introduced in the value of g when this equation is used is less than six thousandths of one per cent. This error is usually less than that introduced by unavoidable experimental errors in measuring T and l. From the fact that the mass of the bob does not enter (109), this equation shows that the period of vibration of a simple pendulum is independent of the mass of the bob.

A pendulum which makes one half of one complete vibration in one second of time is called a *seconds pendulum.*

118. Sympathetic Vibration.—Consider a number of pendulums of different lengths, (Fig. 82), hung from the same support. When one of the pendulums is set into oscillation, all the others will receive slight impulses every time the moving pendulum swings back and forth. The impulses received by any pendulum having the same length as the moving one—*i. e.* the same period—will be always in such a direction as to increase its motion. The other pendulums of periods different from the moving one will receive impulses; but some of the impulses will tend to set the pendulums into motion while others will tend to set them into motion in the opposite direc-

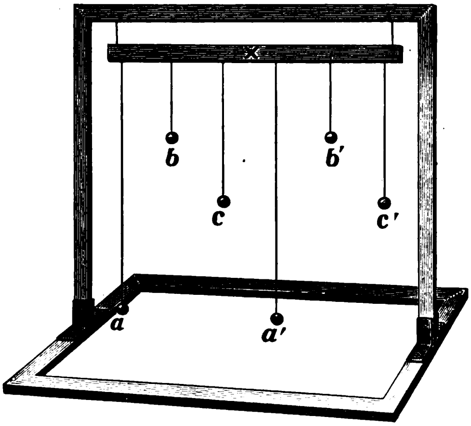

Fig. 82.

tion. The result is that these pendulums are not set into appreciable oscillation. This phenomenon is called *sympathetic vibration* or *resonance.*

It must be noted that as one pendulum receives energy, another one loses energy. The one originally in motion will come to rest when the one that has gained motion is swinging with its maximum amplitude. Neglecting frictional losses, the total energy of the system is constant.

A bridge or other engineering structure has a definite natural period of vibration. If this should happen to be about the same as the step of a trotting horse, a bridge would be dangerously strained by horses trotting across it. Soldiers are required to break step when crossing a bridge. Well built grand stands have failed under the strain of sympathetic vibration induced by the regular stamping of spectators at foot ball games.

PROBLEMS.

1.—The pendulum of a certain clock is 100 cm. long. It is intended to beat seconds. Will it gain or lose at a place where the acceleration due to gravity is 980 cm. per sec. per sec? How much will it gain or lose in 24 hours? [Loses 313.6 sec.]

2.—If the accelerattion due to gravity at the surface of the sun is 28 times as great as at the surface of the earth, what would be the length of a seconds pendulum on the sun? [91.1 ft.]

3.—What is the value of *g* where a seconds pendulum is 40 in. long? How much would the same pendulum lose per hr. at a place where *g* is 32.2 ft. per sec. in a sec? [32.9 ft. per sec. per sec.; 38.3 beats.]

4.—A simple seconds pendulum is lengthened by heat by 0.05 in. Find the number of sec. it will lose in 24 hr. [5.55 sec.]

5.—A pendulum which oscillates in 1 sec. at the equator would gain 5 min. during 24 hr. at the poles. Compare the accelerations due to gravity at the two places. [$g_p : g_e = (289)^2 : (288)^2$.]

6.—A body of mass 1 gm. is moving with simple harmonic motion of translation. If its greatest speed is 20 π cm. per sec., and the amplitude of the oscillation is 10 cm., find the period of oscillation and the force drawing it toward the position of equilibrium when the body is at one end of a swing. [1 sec.; $40 \pi^2$ dynes.]

7.—Find how many seconds per hour a pendulum clock will gain or lose when carried by a balloon ascending with a constant acceleration of 1 ft. per sec. per sec. [56.]

8.—A small heavy ball suspended by a fine thread 6 ft. 4½ in. long was found to make 151 oscillations in 3 min. 31 sec. Find the acceleration due to gravity. [32.2 ft. per sec. per sec.]

§4. *Wave Motion.*

119. Waves.—A motion which goes through the same series of movements at regularly recurring intervals is called *periodic.* A periodic disturbance which is handed on successively from one portion of a medium to another is called a *wave motion.* The sort of wave motion easiest to imagine is that due to a simple harmonic vibration of the particles composing a medium such that each particle performs its vibration slightly later than the particles on one side of it and slightly sooner than the particles to the other side of it.

120. Transverse Wave Motion.—Imagine a long piece of rubber tubing, *AZ*, to be laid in a straight east and west line on a frictionless horizontal plane, and let the west end of the tube, *A*, (Fig. 83), be moved back and forth in a north and south line with a simple harmonic motion. The motion of *A* will cause a neighboring small part of the tube, *B*, to move in a similar manner; but since the tube is not rigid, *B* will perform its motion slightly later than *A*. Thus when *A* has just reached the northernmost point of its path, *B* will not yet have quite reached the northernmost point of its path; and by the time that *B* does reach that position, *A* will already be a short distance on its way back, and so on. In like manner the motion of *B* will cause a neighboring small part of the tube, *C*, to perform a simple harmonic motion of the same period and amplitude, but always lagging slightly behind *B*; *C* will in turn cause another small part of the tube to vibrate; that another, and so on. This sort of motion is illustrated in Fig. 83, in which the north to south dotted lines indicate the paths over which the particles at *A*, *B*, *C*, and *D*, respectively, move back and forth; the full curve indicates the position of the

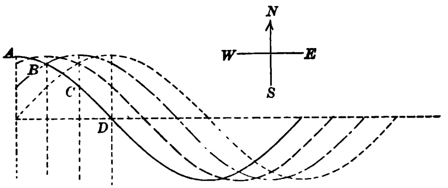

Fig. 83.

tube after the end *A* has made nine twelfths of a vibration; the dashed curve its position after *A* has made ten twelfths of a complete vibration; the dashed and dotted curve its position after *A* has made eleven twelfths of a vibration.This shows that as each particular part of the tube moves back and forth with harmonic motion, there passes to the east along the tube a wave motion. The particular sort of wave motion in which the small parts of the vibrating substance move back and forth in a direction normal to that in which the waves advance is called *transverse wave motion.*

Let Fig. 84 represent, at some particular instant, the position of a part of a length of rubber tubing along which there is passing to the right a transverse wave. The arrows indicate the directions and magnitudes of the instantaneous velocities of the small parts of the tube. Small parts at *a* and *a′* which are in their position of greatest displacement in one direction are said to be at the *crests* of waves, while small parts at *g* and *g′* which are in their positions of greatest displacement in the opposite direction are said to be at the *troughs* of waves. In any wave motion, the distance between two points in consecutive waves which are moving in the same direction with the same speed is called a *wave length.* Thus, each of the dotted lines *aa′*, *bb′ cc′*, represents one wave length.

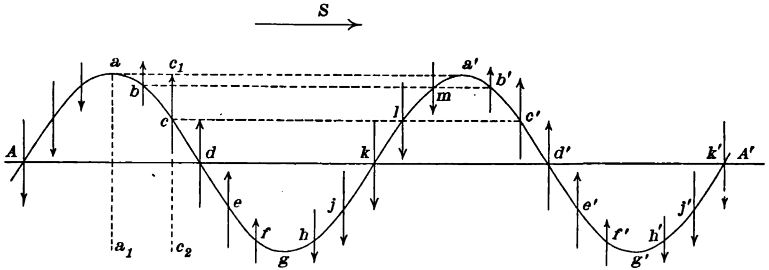

Fig. 84.

121. Longitudinal Wave Motion.—If one end of a long straight piece of rubber tubing be displaced in the direction of its length, then the various small parts of the tube throughout its length will also be successively displaced in the same direction. And if one end be given a simple harmonic motion in the direction of the length of the tube, then the next small part will also execute a simple harmonic motion lagging slightly behind the motion of the end. This second small part will then cause the third also to vibrate, but to vibrate a little later, the third a fourth, and so on. Thus a state will soon be reached in which (*a*), each small part of the tube is moving back and forth with simple harmonic motion in the direction in which the waves are moving, and (*b*) the motion of each small part is performed slightly later than that of its neighbor on one side and slightly sooner than that of its neighbor on the other side. That sort of wave motion in which the direction of vibration of the particles composing the medium is the same as the direction of propagation of the disturbance through the medium is called *longitudinal wave motion.*

On a straight piece of rubber tubing, at rest, let a series

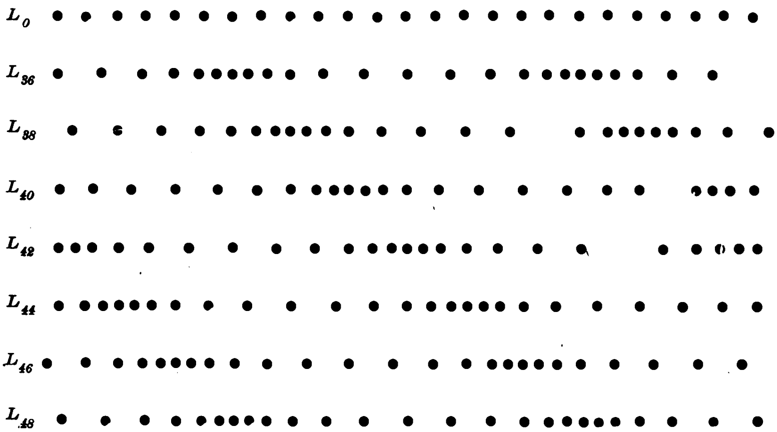

Fig. 85.

of round spots be painted at equal distances apart throughout its length. Now cause a longitudinal wave motion to pass along the tube, and let $12t$ denote the period of the simple harmonic motion which each of these spots executes. In Fig. 85, the line L_0 shows the positions of the spots when no wave is passing, the line L_{36} their position $36t$ seconds after the first spot goes through its position of equilibrium, etc. In this figure it will be noticed that there are places where the spots on the tube are crowded closely together and other places where they are spread apart. A region where the crowding together is a maximum is called a *condensation*; a point where the spreading apart is a maximum is called a *rarefaction.* It will further be noticed that two condensations or two rarefactions remain always the same distance apart, and that, although the particular spots merely move back and forth, yet the condensations and rarefactions move forward.

In Fig. 86 the consecutive spots on the rubber tube are numbered in order. In line L_{36}, the direction and size of the arrowheads represent the direction and magnitude of the velocity of the spots to which they are attached $36t$ seconds after the first spot was in its position of equilibrium. At this instant spots *7* and *19* are in consecutive waves and are moving with the same speed in the same direction. The distance between them is, therefore, one wave length. Similarly, the distance from *8* to *20*, from *9* to *21*, etc., are wave lengths.

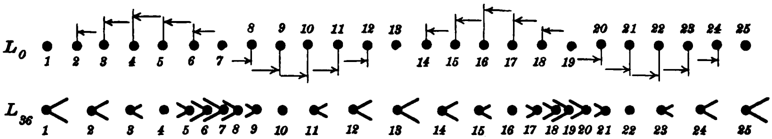

Fig. 86.

122. Speed of a Wave Motion.—In Fig. 84 we see that during the time required for the spot at a to execute a complete vibration, *i. e.* to move to a_1 and back, the spot at c has moved to c_1, thence to c_2, and back to c; and similarly every other spot on the tube has returned to the position it has in the figure. That is, every spot makes one complete vibration while the wave advances one wave length.

In Fig. 85 we see that while time increases from $36t$ to $48t$, *i. e.* during the time required for each spot to execute a complete vibration, the wave moves forward one wave length. If then, a small part of any substance which is transmitting either transverse or longitudinal waves executes a simple harmonic motion of period T, the wave advances one wave length in every T units of time. That is, the speed, s, with which the wave advances is given by

$$s = \frac{\lambda}{T}, \tag{110}$$

where λ denotes the wave length of the motion.

The speed of propagation of a wave of given type through any selected medium, is independent of the wave length, but is different for the two types of wave motion. For instance, the speed of all transverse waves in steel is somewhat over three kilometers per second, while the speed of all longitudinal waves in steel is about twice that amount.

123. Flow of Energy.—(*a.*) *With Transverse Waves.* If a transverse wave is passing to the right along a piece of rubber tubing, (Fig. 87), the force which tends to restore any small part of the tube to its position of rest in the line AA' is due to the tension in the tube. Consider a small part of the tube at l. The forces acting upon it are two tensions, one drawing it toward m and another drawing it toward k. These two forces have components perpendicular to the direction of the tube at l, and it is the sum of these components that increases the downward motion of the small part at l. For every point on the tube, the sum of these components acts toward the concave side of the tube as indicated by the arrows in the figure. That is, these forces all tend to make the tube straight.

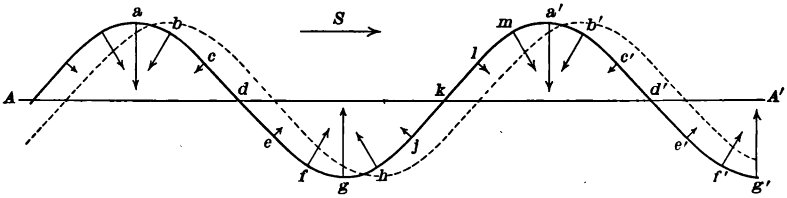

Fig. 87.

In a short time the tube will be in the position of the dotted curve, so that from a to d the parts of the tube are moving against the forces which are acting upon them, from d to g with those forces, from g to k against them, etc. That is, the parts of the tube from a to d are doing work, from d to g are having work done upon them, from g to k are doing work, from k to a' are having work done upon them, and so on. Now the points a, g, a', etc., are momentarily at rest, and are, therefore, neither doing work nor having work done upon them. Whence, at this instant no energy is passing from one side of any of the points a, g, a', etc., to the other side of it. That is, the amount of energy at this instant between a and g is not changing, nor is the amount of energy between g and a' changing. If, then, the amount of energy between a and g is not changing, and if the parts of the tube between a and d are doing work while those between d and g are having work done upon them, it follows that the energy of the parts between a and d is being passed on to the parts between d and g. In the same way it follows that the energy of the parts between g and k is being passed on to the parts between k and a'. Thus, in transverse wave motion, energy is continually moving in the same direction and with the same speed as the advancing wave.

(*b*). *With Longitudinal Waves.* In Fig. 86 the lengths of the horizontal arrows attached to the vertical lines passing through the spots in the line L_0 indicate the displacements of the various spots on the tube at the time when they are in the positions indicated in the line L_{3a}. Since the tube is uniform, and since the spots on the tube are moving back and forth with simple harmonic motions of the same period and amplitude, the forces tending to restore them to the positions which they occupy in the line L_0 are proportional to these displacements (Art. 110). Consequently these arrows also represent the restoring forces acting on the various parts of the tube. From a discussion similar to the one used for transverse waves it can be shown that in a longitudinal wave motion energy is continually moving in the same direction and with the same speed as the advancing wave.

Consequently, *each type of wave motion is accompanied by a flow of energy from one part of a substance to another. The direction of the energy flow is the direction in which the waves move, and the speed of the energy flow is the speed with which the waves move.*

124. Phase.—In the case of vibrations along a horizontal line, displacements, accelerations, and forces directed from the position of equilibrium to the right are counted positive. In the case of vibrations along a vertical line, displacements, etc., from the zero position upwards are counted positive. The *phase* of a harmonically vibrating particle is the fraction of a whole period of vibration which has elapsed since the particle last passed the zero position in the positive direction. Thus, if in Fig. 88, a particle vibrates along the line AE with simple harmonic motion, and if at intervals of one eighth the period of vibration the particle is successively in the position C, D, E, D, C, B, A, B, etc., then when moving in the positive direction the phase of the particle at C is zero, when at D it is $\frac{1}{8}$, when at E it is $\frac{1}{4}$, when moving in the opposite direction the phase at D is $\frac{3}{8}$, when at C it is $\frac{1}{2}$, etc.

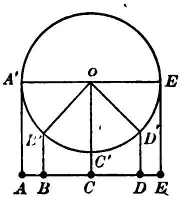

Fig. 88.

Or, since the projection, on any straight line, of a point moving with uniform speed in the circumference of a circle moves with simple harmonic motion, (Art. 111), phase may also be expressed as the angle which the radius passing through the point in the circle of reference makes with the positive direction of the path of the vibrating particle. Thus, in the above figure, when moving in the positive direction the phase of the particle at D is $\frac{1}{4}\pi$ radians or 45°, when at E it is $\frac{1}{2}\pi$ radians or 90°, when moving in the negative direction the phase of the particle at D is $\frac{3}{4}\pi$ radians or 135°, etc.

Two particles differing in phase by zero or by any multiple of 2π radians are said to be in the *same phase*. Thus in Fig. 84, a and a' are in the same phase, b and b' are in the same phase, etc. Two particles differing in phase by π radians are said to be in *opposite phases*. Thus in Fig. 84, a and g are in opposite phases; and in line L_{36} (Fig. 86) *5* and *11*, and again *7* and *13*, are in opposite phases.

125. Interference of Wave Motions.—Let three similar pieces of uniform rubber tubing, AB, $A'B$, BC, be connected as shown in Fig. 89 to the ends of a rod AA' capable of oscillation about a horizontal axis. If the end *A* is moved up and down with simple harmonic motion, waves will be started along the tubes *AB* and *A'B*. The waves in the two tubes will necessarily be of the same period, and if the tubes are of the same length and the distances from *A* and *A'* to the axis of rotation are equal, then the disturbances from *A* and *A'* reaching *B* at any instant will be of equal amplitude and opposite phase. Consequently, the displacement of *B* will equal zero—in other words no wave will be transmitted along *BC*. In general, if two waves of the same period and amplitude meet in opposite phases when traveling in the same direction, the resultant is zero. This neutralization of the effect of one wave motion by another is called *interference.*

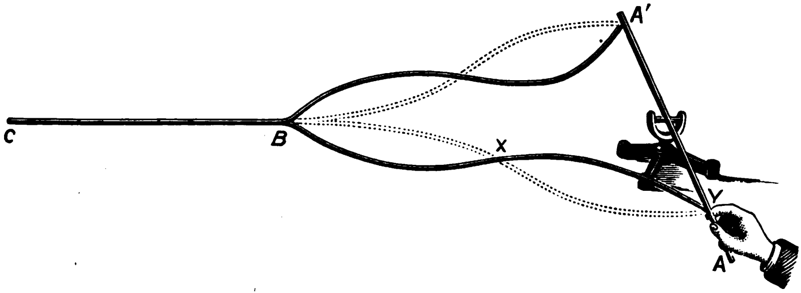

Fig. 89.

Interference is usually not complete. If the amplitudes of the two waves reaching *B* were not equal, then the resultant displacement of *B* at every instant would not be zero, and this point would vibrate with an amplitude equal to the algebraic sum of the amplitudes of the two waves. In this case a wave would be transmitted along *BC*.

Interference is the distinguishing characteristic of wave motion. Only by wave motion can each of two agents acting alone supply a given point with a flow of energy, but acting together impart to the given point zero energy. Interference effects are easily obtained with sound, light, and electricity. Consequently we conclude that in sound, light, and electricity, energy is propagated by means of wave motion.

126. Standing or "Stationary" Waves.—Consider a long uniform rubber tube, *AB*, which lies in a straight line on a frictionless horizontal plane. Let both ends, *A* and *B*, be given transverse simple harmonic motions of the same period and amplitude. Since the tube is uniform and the periods of the two vibrations are the same, the two waves thereby produced will not only have the same amplitude and period but will also move with the same speed toward the middle of the tube. Let the letter *R* denote the wave which is passing to the right from *A* toward *B*, and the letter *L* denote the wave passing to the left from *B* toward *A*. In Fig. 90, a section of the tube two wave lengths long has been chosen, and the wave form for that particular part of the tube is represented at eight chosen instants by the curve marked *W*. The curve marked *R* is in each case the form the rubber tube would have if the tube were affected only by the wave advancing to the right, and the curve marked *L* is the form the tube would have if affected only by the wave advancing to the left. Since the displacement of any small part of the tube at any instant is the result of its displacement due to *R* and that due to *L*, the resultant wave form produced by the two simultaneous waves is obtained by taking for the ordinate of each point the algebraic sum of the ordinates of the two component curves, *R* and *L*. In this manner the curve *W* has been constructed for a series of successive instants one eighth of a period apart.

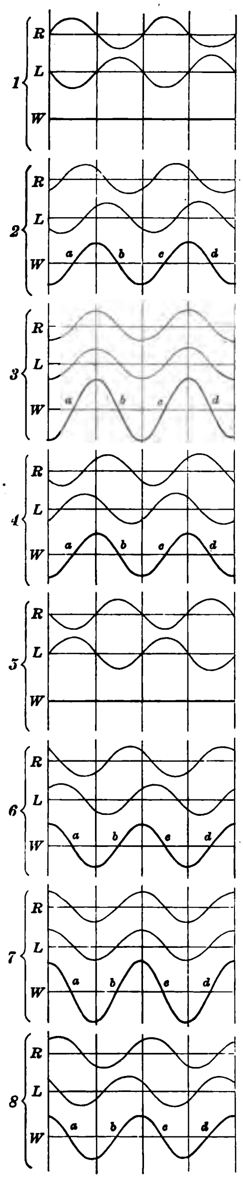

Fig. 90.

At some moment the crests of R will coincide with the troughs of L and complete interference will occur, as shown in (1),(Fig. 90). At the instant one eighth of a period later, R has advanced to the right, and L has advanced to the left, one eighth of a wave length. At this instant the condition is that represented in (2) Fig. 90). It will here be noticed that at certain points, *a, b, c, d,* the two component displacements are still equal and opposite, but at other points they are not. At the end of a time interval equal to another one eighth period we have the moment for which the curves in (3) are drawn. Here we notice that the same points, *a, b, c, d,* are still undisplaced, while other points are displaced. In the same manner we can follow through the other parts of the figure, and as we do so we find that the points *a, b, c, d,* etc., a half wave length apart are never displaced, and that the parts of the tube between them swing out first on one side and then on the other.

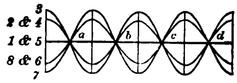

Fig. 91.

In Fig. 91 the W curves of the previous figure are superposed, so that the eight curves in Fig. 91 indicate the positions of the tube at the end of successive eighths of a period.

Throughout the above reasoning transverse waves have been considered. Proceeding in a similar manner it can be shown that if longitudinal wave motions of the same period, amplitude, and speed, meet when moving in opposite directions, there are points a half wave length apart which are never displaced and each of which is alternately in a state of compression and of rarefaction.

The motion which results from the composition of two waves of the same type, period, and amplitude, moving in opposite directions, is called a *standing* or *stationary* wave motion. The points, a half wave length apart, which remain always at rest are called *nodes.* The points midway between the nodes, where the greatest displacements occur, are called *loops* or *ventral segments.*

It is to be noticed that, since the two component wave motions are conveying energy at the same rate in opposite directions, a standing wave effects no continuous transfer of energy from one place to another. In the experiment adduced to illustrate interference, (Fig. 89), the rubber tubes *AB* and *A'B* are in standing wave motion, and there is no flow of energy into the tube *BC*.

127. Polarization.—Imagine a long horizontal rubber tube, one end fastened to the wall and the other end held in the hand. If the free end be moved back and forth in any straight line normal to the length of the tube, each small part of the tube will, in succession, move back and forth in a direction parallel to the initial displacement. A transverse wave motion in which the paths of the small parts of the medium all lie in the same plane, or in parallel planes, is called a *plane polarized wave motion.* If, however, the free end of the tube is moved with uniform speed in the circumference of a circle, a neighboring small part of the tube will be caused to move a little later in a similar manner, this will cause the next part to move in the same way, and so on. Thus, there will advance along the tube a wave motion in which each small part of the tube moves in the circumference of a circle. A wave motion in which the small parts of the medium move in circular paths in parallel planes is called a *circularly polarized wave motion.* In a similar manner *elliptically polarized wave motion* could be produced.

If an observer standing at one end of the tube looked along its axis and saw the tube as a straight line, a circle, or an ellipse, and knew that it was vibrating in some manner, he would be certain that the vibrations were transverse to the length of the tube. If, however, the tube were vibrating, but to the observer appeared as a point, he would be certain the vibrations were in the direction of the length of the tube, that is, were longitudinal.

Let the tube just considered pass through two devices each of which will allow the tube to move freely in the direction of its length, and in *one* direction normal to its length. In each of the apparatus *P* and *A* illustrated in the figure, the tube passes through a loosely fitting ring *R* supported by a light frame *BCR* capable of motion about *BC* as an axis. Thus, the motion of the ring *R* is limited to a line normal to the plane of this frame. When at rest the ring *R* is kept at the center of the circular casting *D* by means of a small spring. The frame *BCR* is attached to this casting, and the latter is so mounted that it can be rotated about the tube as an axis and clamped in any position.

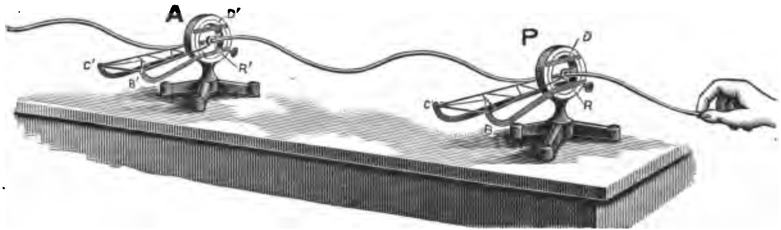

Fig. 92.

Let the apparatus *P* be so arranged that the ring through which the tube passes can move only in a vertical direction. If now, the free end of the tube be vibrated up and down, a transverse wave will advance along the tube and pass without change through *P*; but if the first end of the tube be vibrated in a horizontal plane, no wave will be transmitted past *P*. If the free end be given a transverse vibration in any other direction, the vertical component of the tube's motion will be transmitted past *P*. For instance, if the free end of the tube be given a circular motion in a plane normal to the length of the tube, the tube will be given a motion which will advance to *P* in the form of a helix; but as the ring about the tube at *P* can move only in a vertical line, the wave emerging from *P* will be plane polarized in the vertical plane. Any device that will produce plane polarized waves is called a *polarizer*.

If the instrument *A* be so arranged that its ring can move only in the horizontal direction, no wave that is transmitted by *P* can be transmitted by *A*. If, however, the planes of vibration of the rings of *P* and *A* are not at right angles, then the component, in the direction of motion of the ring of *A*, of the polarized wave incident on it, will be transmitted. A device serving as *A* to detect plane polarized waves is called an *analyzer*. An instrument that can be used as a polarizer can be used as an analyzer, and conversely.

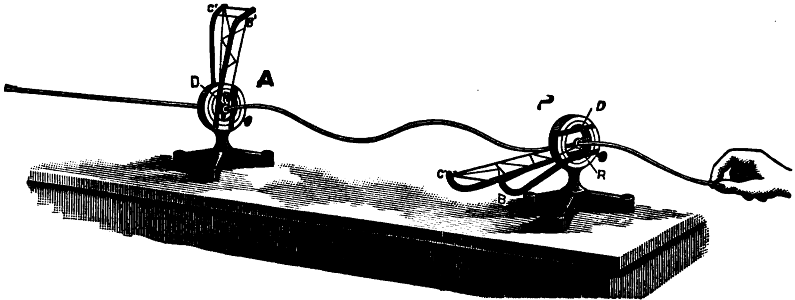

Fig. 93.

The transmission of *longitudinal* waves along the tube is quite unaffected by the planes of vibration of the rings *R* and *R'* of *P* and *A*.

Polarization is the distinguishing characteristic of transverse wave motion. Whenever it can be shown that any wave motion can be polarized, then it is certain that in this wave motion the direction of the vibration is transverse to the direction of propagation of the disturbance. Light and electrical waves can be polarized while sound waves cannot. Therefore we know that the wave motions by which luminous and electrical energy are propagated are transverse, while that of sound is longitudinal.

128. In the preceding paragraphs the consideration of wave motion has been limited to the very special case of vibrations of material particles composing a medium which has practically but one dimension. It should be remarked, however, that wave motion is not restricted either to the vibration of matter, or to the propagation of a disturbance along a single line. Matter is not necessary for the transmission of either luminous or electrical effects. Sound waves proceed in all direction from a sound source.

CHAPTER XI.

STATICS OF FLUIDS.

129. Definitions.—All bodies yield gradually to forces which tend to change their form. The property of a body in virtue of which time is required to change its form is called *viscosity.* A body possessing small viscosity is said to be *mobile.* The resistance which a body offers to a change of form is called *rigidity.* The property of changing form under continuous stress with the development of but small reaction is called *plasticity.*

A body possessing considerable rigidity is called a *solid,* while a body possessing little rigidity is called a *fluid.* A *perfect solid* is a body which suffers no distortion when under the action of a stress. A *perfect fluid* is a body that may be deformed by any force, however small. Fluids are divided into liquids and gases. A solid has a shape and a volume of its own. A liquid has a volume of its own but takes up the shape of whatever contains it. A gas has neither volume nor shape of its own.

By the *pressure,* P, of a fluid at a given point is meant the force, F, exerted by the fluid per unit of area, A, in the specified region. That is,

$$P = \frac{F}{A} \cdot \tag{111}$$

130. General Properties of Fluids.—From the above fundamental characteristic of fluids can be deduced several general properties.

(*a*). *In case of a fluid at rest, the force it exerts on any surface in contact with it must be normal to the surface.* If this were not so, there would be a component of the force parallel to the surface. and this would cause the fluid to move.

Conversely, *at any point, the surface of a fluid at rest must be normal to the resultant force acting at the assigned* *point.* For example, in the case of a liquid, if its weight be the only force acting upon it, the free surface of the liquid will be horizontal.

If a locomotive is moving with a positive acceleration a, any particle of water of mass m in the free surface in the boiler is acted upon by an effective force in the direction of motion equal to ma. This particle is also acted upon by its weight, mg, vertically downward, and the reaction R normal to the surface. If the effective force be considered to be reversed in direction, then d'Alembert's principle (Art. 81) shows that the laws of equilibrium (Art. 42) can be applied. Thus, resolving forces vertically and horizontally,

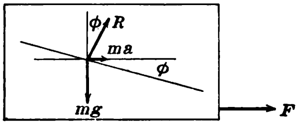

Fig. 94.

$$R \cos \Phi - mg = 0.$$
$$R \sin \Phi - ma = 0.$$

On eliminating from these equations the ratio of R to m, the inclination of the free surface of the water in the boiler to the horizontal plane is found to be

$$\Phi = \tan^{-1} \frac{a}{g}.$$

This shows that during the time the speed of a locomotive is changing, the height of water in the water gage does not give a correct indication of the amount of water in the boiler.

(*b*). *At any point within a fluid at rest, the pressure is equal in all directions.* This can be proven as follows. Consider any small volume of the fluid, for example a small right prism *abce*, Fig. 95. Call the area of the base *bcdf*, A_1; the resultant force acting upon it, F_1; and the pressure acting upon it, P_1. Represent the corresponding quantities for the faces *acde*, *abfe*, *abc*, and *efd* by the symbols A_2, A_3, A_4, A_5, and F_2, F_3, F_4, F_5, and P_2, P_3, P_4, P_5, respectively.

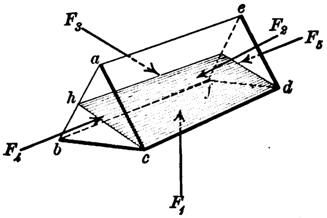

Fig. 95.

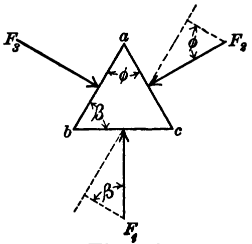

Fig. 96.

Since the fluid is at rest, the sum of the components in any direction of all the forces acting on the element equals zero. Resolving forces in the direction *ab*, we have (Fig. 96),

$$P_1 A_1 \sin abc = P_2 A_2 \sin bac,$$

or

$$P_1 \left[(bc)(cd)\frac{ch}{bc} \right] = P_2\left[(ac)\ (cd)\ \frac{ch}{ac} \right].$$

Whence, $$P_1 = P_2.$$

In a similar manner it can be shown that

$$P_2 = P_3.$$

Consequently $$P_1 = P_2 = P_3.$$

Since the element of volume considered may be taken as small as is desired so that its weight can be neglected, and its faces in any direction whatever, the proposition has been proved.

It should be noted that $(cd)(ch)$ = area *hd* = projection of the faces *bcdf* and *acde* on a plane normal to the face *abfe*. Whatever the shape of the little element of volume taken, the sum of the projections of all the faces on any plane must balance as in the above example.

(*c*). *If a fluid completely filling a vessel is free from the influence of gravity and other external forces, it will exert the same pressure at all points throughout the fluid and at all points against the walls of the vessel.* This is because if there were a greater pressure at one region than at another, the fluid would immediately flow until the pressures became the same throughout.

The preceding law, (*c*), governs the action of the hydrostatic press. This machine consists essentially of two cylinders of different diameters, closed by pistons, and connected by a tube. The cylinders and the connecting tube are filled with liquid. Let the areas of cross-section of the pistons be denoted by A_1 and A_2 respectively. If a force F_1 be applied to the piston having the area A_1, a force F_2 must be applied to the piston having the area A_2 in order to balance the force F_1. Since the pressure is transmitted equally throughout the liquid, the pressures on the two pistons are equal. That is, the force F_2 is given by the equation

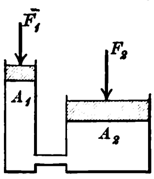

Fig. 97.

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}.$$

By this device a small force applied to a small piston can produce a great force on a large piston.

In the special case of a liquid in a vessel open to the air, there is a uniform pressure on the free surface of the liquid due to the weight of the air. This produces a uniform pressure throughout the liquid and against the walls of the vessel, exactly as though the mouth of the vessel were closed with a piston acted upon by a pressure equal to that of the atmosphere.

131. Fluid Pressure due to Weight.—In the case of a fluid at rest, the pressure at two points lying in the same horizontal plane will first be considered. Let A and B, (Fig. 98), lying in the same horizontal plane, be the two points under consideration. Consider the forces acting upon a cylindrical element of the liquid extending from A to B. Suppose the ends of this element are normal to its axis. Acting on this element are, (a), the oppositely directed thrusts on the end faces; (b), the thrusts on the curved surface; (c), the weight of the fluid composing the element. The only forces acting in the direction of the axis of the element are the thrusts on the ends at A and B. Since the liquid is at rest, these forces must be equal and oppositely directed. Since the areas of the two end faces are equal, it follows that the fluid pressure at A equals that at B. Consequently, in the case of a fluid at rest, the pressures are equal at all points in the same horizontal plane.

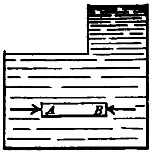

Fig. 98.

The pressure at any point in a fluid depends upon the weight of the fluid above it. The pressure of a fluid at any point will now be determined. If there is no force acting upon the fluid except that due to its own weight, the total force acting upon any horizontal surface of area A at a distance h below the upper surface of the fluid is equal to the weight of a column of the fluid of height h and area of cross section A. If the mean density of the fluid be denoted by D, the mass of this volume is DAh, and its weight is $DgAh$. Whence, the pressure downward at the given point, due to the weight of the fluid, is, by (111),

$$P = Dgh. \tag{112}$$

If, in addition to the weight of the fluid there is acting on its surface a pressure of magnitude P_1, the pressure at any point in the fluid at a distance h below the upper surface is

$$P' = P + P_1 = Dgh + P_1. \qquad (113)$$

This pressure is independent of the shape of the containing vessel.

132. Atmospheric Pressure. The Barometer.—If a long tube, AB, closed at one end, be filled with liquid and then, without any air being admitted, the open end be submerged in a reservoir of the same liquid, the column of liquid in the tube will be held in equilibrium under the action of the weight of the liquid in the tube, and the pressure on the surface of the liquid in the reservoir due to the weight of the atmosphere. It was shown in the previous article that the pressure is the same at all points of any specified horizontal plane. The only pressure on the free surface of the liquid in the reservoir is that due to the weight of the atmosphere. The pressure at the same level inside the tube is Dgh, where D is the density of the liquid and h is the height of the free surface of the liquid in the tube above the free surface of the liquid in the reservoir, plus the pressure due to the vapor above the liquid column. Consequently, the pressure of the atmosphere is

$$P = Dgh + p_2, \qquad (114)$$

where p_2 is the very small pressure due to the vapor of the liquid above the column of liquid in the tube. This pressure, p_2, is so small that it is usually negligible compared with Dgh.

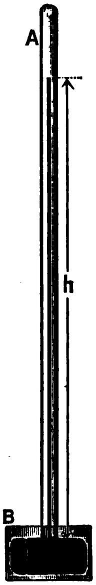

Fig. 99.

The apparatus above described is called a *barometer*. The length, h, of the vertical column of liquid supported by the atmosphere is called the *barometric height* at the given place at the specified time. The barometric height is different for different places and at different times. At sea-level, at latitude 45°, the mean barometric height is about 76 centimeters of mercury. That is, the atmosphere will support a column of mercury 76 cm. high. Whence the mean atmospheric pressure under these conditions is

$$\begin{aligned} P &= Dgh = 13.596 \times 980 \times 76 \text{ dynes per sq. cm.} \\ &= 1.013 \times 10^6 \text{ dynes per sq. cm.} \\ &= 14.7 \text{ pounds' weight per sq. in.} \end{aligned}$$

133. The Open Manometer.—By balancing the pressure of a gas against the pressure due to the weight of a column of liquid of known density, the pressure of the gas can be determined. Let an open glass tube, AB, (Fig. 100,) containing some liquid (*e. g.* water or mercury) be bent as in the figure and joined by one end to the vessel, R, containing the gas whose pressure is sought. If the pressure of gas in the reservoir is different from the pressure of the air outside, there will be a difference in level of the liquid in the two arms of the tube. If the atmospheric pressure be denoted by P, the density of the liquid by D, and the difference in level of the free surfaces of the liquid in the two tubes by h, then the pressure of the gas in the reservoir is

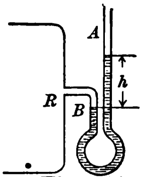

Fig. 100.

$$P' = Dgh + P. \qquad (115)$$

This instrument is called an *open manometer.*

134. The Common Pump.—The common lifting pump, (Fig. 101), consists of a cylinder supplied with an upward opening valve V and a closely fitting piston supplied with one or more upward opening valves vv. While the piston is being raised, the valves vv close, the pressure of the air in the lower part of the cylinder is diminished below the atmospheric pressure, the valve V opens, and the pressure of the atmosphere on the surface of the water into which dips the end of the supply pipe forces water up into the cylinder. If the piston be now lowered, the valve V closes, and the valves vv open, thus allowing the water in the cylinder to flow above the piston. On again raising the piston, the water above it flows through the outlet O, and more water is forced up into the low pressure space below the piston.

The maximum height, h', to which a perfect lifting pump can raise a liquid of density D' is given by the condition that the atmospheric pressure is

$$P = D'gh'.$$

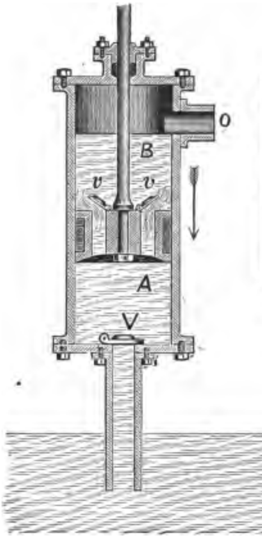

Fig. 101.

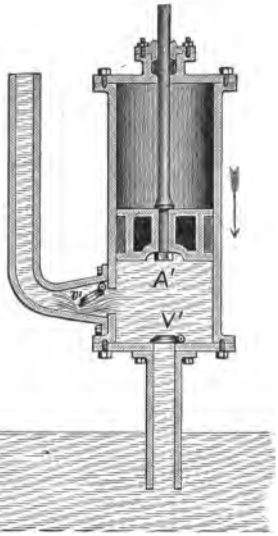

Fig. 102.

In the case of water, D' is 62.4 pounds per cubic foot. Whence, in this case

$$14.7\,g = \frac{62.4}{1728}\,gh' \text{ poundals per square inch.}$$

Whence $h' = 407$ inches $= 34$ feet (nearly.)

The action of the common force pump will be made clear by a study of Fig. 102.

135. Upward Pressure in Fluids. Archimedes' Principle.—Imagine any portion, X, of a fluid at rest to be separated from the remainder by an imaginary enveloping film entirely submerged below the free surface of the fluid. Since the entire mass of fluid is at rest, the portion X must be acted upon by a force whose magnitude is equal and whose direction is opposite to its weight. This force is due to the pressure of the fluid Y being greater on the lower parts of the portion X than upon the upper parts. If the fluid within the imaginary film be replaced by another body having the same shape and size, the pressure on the outside of the film will not be altered. Whence, *any body immersed in a fluid at rest loses in*

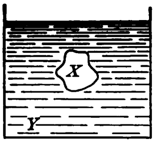

Fig. 103.

weight an amount equal to the weight of the fluid displaced by the body. This is called Archimedes' Principle.

If the volume of the immersed body be denoted by v, its weight in vacuo by W, its weight when immersed in a fluid of density D' by W', then Archimedes' principle can be stated in the analytical form

$$W - W' = D'vg. \tag{116}$$

Archimedes' principle shows that a body lighter than an equal volume of the fluid in which it is immersed will be forced upward by the surrounding fluid. For example, the envelope of a balloon distended with hydrogen, coal gas, or hot air will rise in the atmosphere. Since the sole object of the gas in the balloon is to displace a large volume of air, a gas should be selected that has the greatest volume possible per unit of mass.

If the weight of a body equals the weight of the fluid displaced, the body will be in equilibrium in the fluid. A body floating in a liquid is an example of this case. An iron ship floats, not because the materials entering into its construction have a smaller density than water, but for the reason that its hollow form causes it to displace a mass of water equal to its own mass. A 5000 ton ship is one that displaces 5000 tons of water; that is. one that weighs 5000 tons. The tonnage of a vessel is computed from the volume of that part of its hull below the water line.

136. Determination of Density by Immersion.—Whenever a solid body is of such simple geometrical form that its volume can be computed from its linear dimensions, its density can be obtained directly from the definition, (Art. 82),

$$D = \frac{m}{v}\,. \tag{117}$$

Very often, however, a body is of such irregular shape that its volume cannot be computed from its linear dimensions. In such cases, the principle of Archimedes furnishes a simple indirect method of finding the volume. Thus, consider a body that will sink when placed in a liquid of density D'. If the weight of the body when in vacuo be denoted by W, and its weight when immersed in a liquid of density D' be denoted by W', (116) gives for the volume of the body

$$v = \frac{W - W'}{D'\,g}.$$

Now since the error that would be introduced in the final result by neglecting the small difference between the weight of the body in vacuo and its weight in air is less than the error usually introduced by the weighing, it is customary to let W be the weight of the body in air, instead of its weight in vacuo.

On substituting the above value of v in (117), we have

$$D = \frac{m}{v} = \frac{mgD'}{W - W'} = \frac{WD'}{W - W'}. \qquad (118)$$

Since, by definition, (Art. 82), specific gravity is the ratio of the density of a body to the maximum density of water,

$$\text{Sp. Gr.} = \frac{WD'}{(W - W')D_2}, \qquad (119)$$

where D_2, the maximum density of water, has the value 1 gram per c.c., or 62.4 + pounds per cu. ft.

In the case of a body of so small density that it floats in the liquid, the body must be submerged by means of a sinker. The method of determining the density in this case is left as an exercise for the student.

137. From Archimedes' principle, the mass of liquid displaced by a body floating in it is equal to the mass of the floating body. But the mass of liquid displaced equals the product of the density of the liquid and the volume of the immersed portion of the floating body. Consequently, if a body of given mass be allowed to float in a liquid, the density of the liquid can be determined from the volume of the submerged portion.

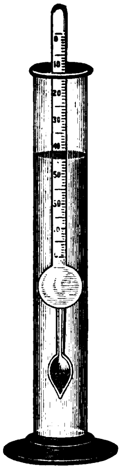

Fig. 104.

An instrument depending upon this principle is called an areometer or hydrometer. The ordinary hydrometer, (Fig. 104), consists of a graduated glass stem to which is attached a weighted glass bulb. By placing the instrument into two or more liquids of known densities the scale on the stem can be calibrated to read specific gravities directly.

PROBLEMS.

1.—What is the pressure at a depth of 50 ft. below the surface of the sea, the specific gravity of sea water being 1.025? [3198 lbs. wt. per sq. ft. plus atmospheric pressure.]

2.—Mercury is poured into a vessel until the layer is 10 cm. deep, and then water is added until the depth of the water column is 75 cm. Determine the pressure on the base in dynes per sq. cm. [1,217,700 dynes per sq. cm.]

3.—Determine the available water-pressure (in lb. wt. per sq. in.) in a laboratory which is supplied from a tank at a height of 40 ft. [17.3 lbs. wt. per sq. in.]

4.—The lever of a hydraulic press gives a mechanical advantage of 6; the sectional area of the smaller plunger is 0.5 sq. in. and that of the larger plunger 15 sq. in. A 56 lb. mass is hung from the handle. What weight will the larger plunger sustain? [10,080 lbs. wt.]

5.—If the diameters of the pistons of a hydraulic press are in the ratio of 10 to 1, and a force of 14 lb. wt., applied to the handle produces a pressure of 2240 lb. wt. on the larger, what is the ratio of the arms of the lever used for working the piston? [8:5.]

6.—The two pistons of a hydraulic press have diameters of a foot and an inch respectively. What is the pressure exerted by the larger piston when a weight of 56 lb. is placed on the smaller one? If the stroke of the smaller piston is 1.5 in., through what distance will the larger piston have moved after ten strokes? [8064 lbs. wt.; 0.104 in.]

7.—If the density of air is 0.001,276 gm. per c.c. and if the top of the bluff just north of the West LaFayette water works is 30 m. above the river level, what will be the difference in the readings of two barometers placed respectively at the river level and on top of the bluff? [2.82 mm.]

8.—If the area of a man's body is 1.5 sq. m., what is the total atmospheric force to which he is subject when the barometer stands at 753 mm.? [15,360 kg. weight.]

9.—If an open mercury manometer is connected to a tap 50 ft. below the surface of water in a tank, what will be the difference in height of the two mercury columns? [3.67 ft.]

10.—What change in the atmospheric pressure is indicated by a fall of one inch in the height of the barometric column? [0.492 lb. wt. per sq. in.]

11.—What is the head of water at a point where the pressure is 50 lb. wt. per sq. in. [115.3 ft.]

12.—An iron body weighing 275 gm. floats in mercury with 0.556 of its volume immersed. Required the volume and density of the body. [36.3 c.c.; 7.56 gms. per c.c.]

13.—A bar of aluminum (Sp. Gr. 2.6) weighs 54.8 gm. in vacuo. What will be the loss of weight when it is weighed in water? [21.1 gm.]

14.—A hollow stopper made of glass of density 2.6 gm. per c.c. weighs 23.4 gm. in air and 3.9 gm. in water. What is the volume oi the internal cavity [10.5 c.c.]

15.—The density of amber is 1.078 gm. per c.c. and that of sea-water is 1.025 gm. per c.c. How far would a piece of amber sink in 1 sec. in the sea, starting from rest? [24 cm.]

16.—The specific gravity of ice is 0.918 and that of sea-water is 1.03. What is the total volume of an iceberg which floats with 700 cubic yards exposed? [6438 cu. yd.]

17.—A solid body floats in water with just half its volume immersed. When it floats in a mixture of equal volumes of water and another liquid one-third of it is immersed. Find the specific gravities of the solid and liquid. [Solid, 0.5; liquid, 2.0.]

18.—A solid which weighs 120 gm. in air is found to weigh 90 gm. in water and 78 gm. in a strong solution of zinc sulphate. What is the specific gravity of the solution? [1.40.]

19.—The volume of a balloon filled with coal gas is 1,000,000 liters and its mass, including the car, is 500 kg. If the density of the air is 1.28 and that of the coal gas is 0.52, both being in grams per liter, find what additional weight the balloon can sustain in the air. [260 kg. wt.]

20.—What mass of lead of density 708 lb. per cu. ft. is required to submerge in water a cork life preserver weighing 10 lb. The specific gravity of cork is 0.24. [34.7 lb.]

21.—A piece of iron when plunged into a vessel filled with water causes 10 gm. to overflow. When placed in a vessel full of mercury it floats, displacing 78 gm. of mercury. Find the mass, volume, and Sp. Gr. of the iron. [78 gm. 10 c.c. 7.8.]

22.—A sphere having a density 0.957 gm. per c.c. and a volume of 168 c.c. floats in a vessel of water. If a layer of oil having a density of 0.892 gm. per c.c. is poured on the water so as to cover the sphere, how much of the latter will be immersed in the water? [101 c.c.]

23.—Find the mean density of a mixture of 8 parts by volume of a substance of density 7, with 19 parts of a substance of density 3. [4.185.]

24.—Find the mean density of a mixture of 8 parts by weight of a substance of density 7, with 19 parts of a substance of density 3. [3.611.]

25.—Two parts by volume of a liquid of density 0.8 are mixed with 7 of water, and the mixture shrinks in the ratio of 21 to 20. Find the Sp. Gr. of the mixture. [1.0033.]

26.—A body weighing 1030 gm. in air weighs 905 gms. in water at 4° C. What would it weigh in vacuum? Sp. Gr. of air is 0.0013 and of the brass weights used in weighing 8.5. [1030.005 gm.]

CHAPTER XII.

KINETICS OF FLUIDS.

§1. *The Uniform Flow of Fluids through Tubes.*

138. The Fundamental Equation.—In the present chapter the general equation of the flow of fluids through tubes will be derived, and then this equation will be applied to the study of a few special problems such as the action of the jet pump, the siphon, and the steam injector. Consider a fluid flowing steadily through a frictionless tube of variable section which it entirely fills. In order to fix the ideas, consider a thin lamina of the fluid at X_1 normal to the axis of the tube, and another similar lamina at X_2. Let the pressure in the first lamina be denoted by P_1, and the pressure in the second lamina by P_2. As the fluid flows, the first lamina moves through a distance x_1 to the position X_1' with a speed s_1, while the second lamina moves through a distance x_2 to the position X_2' with a speed s_2. The distances x_1 and x_2 are so short that s_1 and s_2 may be taken to be constant quantities. Let the volume between X_1 and X_1' be denoted by v_1, and the volume between X_2 and X_2' be denoted by v_2. Let the areas of the laminae X_1 and X_2 be A_1 and A_2 respectively. When between the laminae X_1 and X_2, the fluid may be referred to as in the *first* position, and when in the position X_1' X_2' as being in the *second* position.

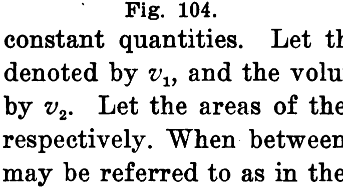

Fig. 104.

se the principle of the conservation of
total energy of the fluid when in the
total energy when in the first
pon the fluid during its mo-

tion to the second position, W_{upon}, minus the work done *by* the fluid during its motion to the second position, W_{by}. That is,

$$E_2 = E_1 + W_{upon} - W_{by}. \qquad (120)$$

The total energy of a given portion of the fluid in any position equals the sum of its potential, kinetic, and internal energies when in the assigned position. When in the first position, the fluid included between the laminae X_1 and X_2 has a potential energy due to its position above the datum plane equal to the sum of the potential energies of the portion included between X_1 and X_1' and the portion included between X_1' and X_2 If the mass of the first portion be called m_1, and if its center of mass is at a height h_1 from the datum plane, then the potential energy of this portion equals m_1gh_1. Denoting the potential energy of the portion included between X' and X_2 by E_p, we have for the total potential energy of the liquid included between X_1 and X_2 when in the first position,

$$m_1gh_1 + E_p.$$

Similarly the kinetic energy of the same portion of fluid in the same position as above, equals

$$\tfrac{1}{2}\, m_1s_1^2 + E_k,$$

where E_k is the kinetic energy of the portion included between X_1' and X_2.

Whence, denoting the internal energy of the fluid when in the first position by e_1, we have for the total energy of the fluid included between X_1 and X_2,

$$E_1 = m_1gh_1 + E_p + \tfrac{1}{2}\, m_1\, s_1^2 + E_k + e_1.$$

In precisely the same manner it is found that the total energy of the fluid included between the two laminae when in the second position is

$$E_2 = m_2gh_2 + E_p + \tfrac{1}{2}\, m_2\, s_2^2 + E_k + e_2.$$

Now the work done *upon* this fluid during its motion into the second position equals

$$P_1\, A_1\, x_1 = P_1\, v_1,$$

while the work done *by* the fluid during this motion equals

$$P_2\, A_2\, x_2 = P_2\, v_2.$$

Under the condition of steady flow, the mass m flowing in a given time across any section of the filled tube is a constant quantity. Whence, $m_1 = m_2 = m$.

Consequently, from (120)

$$mgh_1 + \tfrac{1}{2}\, ms_1^2 + e_1 + P_1\, v_1 = mgh_2 + \tfrac{1}{2}\, ms_2^2 + e_2 + P_2\, v_2$$
$$= \text{a constant quantity.} \qquad (121)$$

If the densities of the portions of fluid under the pressures P_1 and P_2 be denoted by D_1 and D_2 respectively, we shall have $v_1 = m \div D_1$ and $v_2 = m \div D_2$. Whence, the above equation assumes the form,

$$gh_1 + \tfrac{1}{2}\, s_1^2 + \frac{e_1}{m} + \frac{P_1}{D_1} = gh_2 + \tfrac{1}{2}\, s_2^2 + \frac{e_2}{m} + \frac{P_2}{D_2}$$
$$= \text{a constant quantity.} \qquad (122)$$

This is the fundamental equation for the uniform flow of fluids through tubes. It applies to both liquids and gases. It is sometimes called Bernouilli's Equation.

---

§2. *The Flow of Incompressible Fluids.*

139. The Flow of Liquids.—Since there is practically no change in the volume of a liquid as its pressure is changed, the fundamental equation of fluid flow (121) becomes in this case,

$$mgh_1 + \tfrac{1}{2}\, ms_1^2 + P_1\, v + e_1 = mgh_2 + \tfrac{1}{2}\, ms_2^2 + P_2\, v + e_2 = \text{const.}$$

If the density of the liquid be denoted by D, this equation can be put into the form

$$h_1 + \frac{s_1^2}{2\,g} + \frac{P_1}{Dg} + \frac{e_1}{mg} = h_2 + \frac{s_2^2}{2\,g} + \frac{P_2}{Dg} + \frac{e_2}{mg} = \text{const.}$$

It is found that so long as the temperature does not change, the internal energy of a liquid is practically the same at all pressures. Consequently the above equations can be written

$$mgh_1 + \tfrac{1}{2}\, ms_1^2 + P_1 v = mgh_2 + \tfrac{1}{2}\, ms_2^2 + P_2 v = \text{const.} \qquad (123)$$

and $$h_1 + \frac{s_1^2}{2\,g} + \frac{P_1}{Dg} = h_2 + \frac{s_2^2}{2\,g} + \frac{P_2}{Dg} = \text{const.} \qquad (124)$$

140. The Speed of Efflux of a Liquid.—Two cases will be considered: First, where the liquid escapes from an orifice in a closed reservoir, as, for example, from a steam boiler; and second, where the liquid escapes from an orifice in an open vessel, as, for example, from a water tank.

Closed Reservoir. Let the surface of the liquid in the reservoir be kept at a constant level, and under a constant pressure P_1. Since the surface is kept at a constant level, its speed $s_1 = 0$. Whence for this case, (123) can be put into the form

$$s_2^2 = \frac{2}{m}(mgh_1 - mgh_2 + P_1 v - P_2 v). \tag{125}$$

If the outlet is on the level of the surface of liquid in the reservoir, $h_2 = h_1$, and

$$s_2^2 = \frac{2}{m}(P_1 v - P_2 v) = \frac{2(P_1 - P_2)}{D}. \tag{126}$$

Open Reservoir. Consider a reservoir kept filled with liquid to a constant level at a height h_1 above the orifice. Since the level of this upper surface is constant, its speed $s_1 = 0$.

If the horizontal datum plane be through the orifice, $h_2 = 0$. Since both the top of the reservoir and the outlet open into the air, $P_1 = P_2$. On substituting these values in either (123) or (124) we have for the speed of efflux from the orifice

$$s_2 = \sqrt{(2gh_1)}. \tag{127}$$

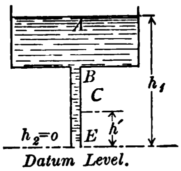

Fig. 106.

In the case of either an open or a closed reservoir, and whatever be the cause of the liquid pressure at the orifice, if this pressure be expressed in terms of the height, h_1, of a column of the liquid that would produce the same pressure, the speed of the outflowing liquid is given by the above equation.

From (127) it will be observed that the speed of efflux of a frictionless liquid from an orifice in an open reservoir is equal to that which would be acquired by a body falling freely from the level of the free liquid surface to the level of the center of the orifice.

141. Lateral Diminution of Pressure.—Let it be required to determine the pressure exerted by a flowing liquid at any point C on the wall of a uniform tube BE, (Fig. 106). Let the height of the point under consideration above the horizontal datum plane drawn through the outlet, E, be denoted by h'. Denote the pressure and the speed of the liquid at the point C by P' and s', respectively. The corresponding quantities at the outlet E will be represented by the symbols h_2, P_2, and s_2. It should be noted that $h_2 = 0$.

Since a liquid is practically incompressible, the volume flowing per second across any cross-section of the tube must be constant. If the areas of cross-section at C and E are represented by A' and A_2, respectively, we shall have

$$A' s' = A_2 s_2. \qquad (128)$$

If the cross-section of the tube is uniform, $A' = A_2$, whence $s' = s_2$. Making in (124) these substitutions, we have

$$h' + \frac{P'}{Dg} = \frac{P_2}{Dg},$$

or

$$P' = P_2 - Dgh'. \qquad (129)$$

Since the orifice opens into the atmosphere, P_2 is the atmospheric pressure. Therefore, the pressure exerted by the liquid at the point C is less than the atmospheric pressure by an amount equal to the pressure that would be exerted at the outlet, E, by a column of the liquid at rest having a height h'.

It thus appears that a moving liquid produces a lateral diminution of pressure at the surface separating it from the surrounding medium. The same effect is also produced by flowing gases.

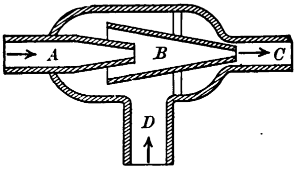

Fig. 107.

The fact that a flowing fluid produces a diminution of pressure at the surface separating it from the surrounding medium finds many practical applications. In the jet-pump, Fig. 79, water at high pressure passes through the jet nozzle A, and escapes through the outlet C. At the throat, B, the flowing water produces a a sufficient diminution of pressure to cause water to rise in the tube *D*. The efficiency of such a pump is from fifty to seventy per cent.

The forced draught of locomotives and marine boilers is produced by a jet of steam from the exhaust nozzle placed in the smoke box. The increased draught of chimneys when a wind is blowing is due to this same cause.

142. If two points in a tube are at the same level, equation (124) shows that where the speed of the flowing liquid is large, the pressure must be small. Equation (128) shows that the speed is greatest where the area of cross-section is smallest. Consequently the pressure is least where the area of cross-section is smallest. The equation giving the value of the diminution of pressure at a constriction can be obtained directly by combining (124) and (128).

143. The Siphon.—A siphon is an inverted U-shaped tube with one limb longer than the other, and is used to make a liquid flow over a barrier. The tube is filled with the liquid, its shorter limb dipped in the basin to be emptied, and its longer limb pointed downward outside of the basin.

Denote the density of the liquid by *D*, and the pressure of the atmosphere by *P*. Then, if the liquid were at rest, the pressure in both branches of the tube at the level of the free liquid surface in the reservoir would be downward and equal to the atmospheric pressure. Whence the pressure at the outlet, X_2, would exceed the atmospheric pressure by the amount Dgh_2. Consequently, the liquid in the siphon will start to flow out of the orifice X_2.

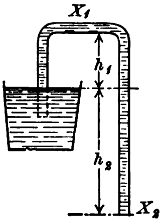

Fig. 108.

The condition necessary for the siphon to continue to operate is now to be determined. Since the flow will cause a lateral diminution of pressure at all parts of the tube, the pressure at any point will be less than when the liquid is at rest. A perfect liquid could not sustain a tension. Therefore if the stream through the tube is to be continuous, the pressure must nowhere be less than zero. That is, the greatest value h_1 can have is such as to cause a pressure at the highest point, X_1, of $P_1 = 0$. From (124) we then have

$$h_1 + \frac{s_1^2}{2g} + 0 = -h_2 + \frac{s_2^2}{2g} + \frac{P_2}{Dg}. \tag{130}$$

Whence, for continuous working, the maximum height, h_1, is

$$h_1 = \frac{s_2^2 - s_1^2}{2g} + \frac{P_2}{Dg} - h_2. \tag{131}$$

Since the orifice opens into the atmosphere, P_2 equals the atmospheric pressure. It should be noted that the term $\frac{P_2}{Dg}$ represents the height of a column of the liquid which would exert on its lower surface a pressure equal to the pressure of the atmosphere. If the tube of the siphon be of uniform bore $s_1 = s_2$. If the liquid be water, $D = 0.0361$ pounds per cubic inch. And since the normal pressure of the atmosphere is 14.7 pounds' weight per square inch,

$$\frac{P_2}{Dg} = \frac{14.7}{0.0361} \text{ inches} = 34 \text{ feet (nearly).}$$

Consequently, for continuous working, no part of a siphon tube of uniform bore can be at a greater distance above the surface of the water in the reservoir than

$$h_1 = (34 - h_2) \text{ feet.} \qquad (132)$$

144. Consider a case in which the area of cross-section of the tube is not uniform. Represent the areas of cross-section at X_1 and X_2 by A_1 and A_2, respectively. Suppose $A_2 = 2A_1$, and $h_2 = 5$ feet. It is required to find the maximum height, h_1, at which the siphon will operate continuously, with water as the liquid to be moved.

Since water is practically incompressible, it follows that in order that the column of water may be continuous, the same volume must pass X_1 and X_2 per second. That is,

$$A_1 \; s' \doteq A_2 \; s_2'.$$

Whence,

$$s_1' = s_2' \frac{A_2}{A_1} = 2s_2'.$$

The value of s_2' can be found by means of (124). Take the surface of the reservoir for the datum plane and also for the section indicated in Art. 138 by X_1. Denote the speed of this section by s_1' and the pressure on its surface by P. Take the outlet X_2 as the second section. The pressure at this section, being due solely to the atmosphere is also P. Since the area of the surface of the reservoir is so great compared with that of the liquid in the tube, its speed s_1', may without sensible error be taken as zero compared with the speed of efflux s'_2. On subsituting these values in (124) we obtain

$$0 + 0 + \frac{P}{Dg} = -5 + \frac{s_2'^2}{2g} + \frac{P}{Dg}.$$

Whence, the speed of the water escaping from the lower end of the siphon is

$$s_2' = 18 \text{ feet per second,}$$

while the speed of the water flowing in the highest part of the siphon is

$$s_1' = 2 \; s'_2 = 36 \text{ feet per second.}$$

On substituting in (131) the above values, we obtain

$$h_1 = \frac{321 - 1284}{64.2} + 34 - 5 \text{ feet}$$

$$= 14 \text{ feet.}$$

That is, if the highest point of this siphon is more than 14 feet above the surface of the water in the reservoir, the stream of water in the siphon will break, and the flow be discontinuous.

§3. *The Flow of Gases.*

145. The Speed of Efflux of a Gas.—Consider a reservoir containing gas under a pressure P_1. Let there be in the reservoir an orifice opening into a space where the pressure has some smaller value P_2. The gas will flow from the reservoir with an initial speed $s_1 = 0$, and escape into the second space with a speed s_2. Consequently, if during the expansion of the gas it neither receives nor loses heat, we shall have from the fundamental equation of the flow of fluids, (121),

$$s_2^2 = \frac{2}{m}(mgh_1 - mgh_2 + e_1 - e_2 + P_1 v_1 - P_2 v_2). \quad (133)$$

But, since the density of a gas is small, if the difference between h_1 and h_2 is not excessive the term $(mgh_1 - mgh_2)$ can without sensible error be neglected. We then have

$$s_2^2 = \frac{2}{m}(e_1 - e_2 + P_1 v_1 - P_2 v_2) \quad \text{approx.} \quad (134)$$

A comparison of (133) and (134) with (125) and (126) shows that for a frictionless orifice, the speed of efflux of a gas exceeds the speed of efflux of a liquid under the same conditions of initial and final pressures by an amount which depends upon the difference between the internal energies of the gas at the two pressures. If a gas expands without receiving heat from some outside source, its temperature falls. This means that some of the internal energy of the gas has been used up in the process of expansion. Consequently in (133)) and (134) e_1 is greater than e_2. In the case of steam, $(e_1 - e_2)$ is such a large quantity that if two similar orifices be made in a steam boiler, one above the water line and one below, the speed of the escaping steam would be over twenty times as great as the speed of the escaping water.

If, during a change in pressure a gas neither receives nor loses heat, then whenever a change in pressure occurs, a change in the same direction will take place in the internal energy.

That is, the magnitude of $(e_1 - e_2)$ increases when $(P_1 - P_2)$ increases, and conversely.

146. The Steam Injector.—By means of the ordinary steam injector, steam from a boiler is used to force feed-water into the same boiler. The operation of the injector involves a number of physical principles, among which are the conservation of energy, the conservation of momentum, and the fact that there is a great difference between the speed of a jet of steam and the speed of a jet of water when the flow of each is due to the same pressure difference.

In the diagram, (Fig. 109), steam from the boiler enters at *X*. As it traverses the steam-nozzle, *a*, the pressure in the surrounding space is reduced as the result of two causes; first, the condensation of some of the steam, and, second, the lateral diminution of pressure due to the moving fluid. For these causes the water rises in the feed-water pipe connected to *Y* and flows into the combining-tube, *bb′*. Here steam and water are mixed and the combined stream passes through the delivery-tube, *c*, toward the boiler connected at *Z*.

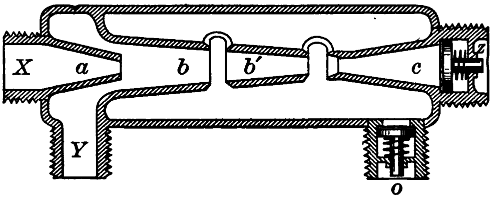

Fig. 109.

In order to make the idea of the action of the injector more concrete, imagine there is in a reservoir an orifice through which water escapes with a speed s. Let a tube conveying water be connected to the orifice. If, from any cause whatever, the speed of the stream toward the reservoir is greater than s, water will flow from the tube into the reservoir. This is due to the greater kinetic energy of the more rapidly moving stream which enables it to exert a greater force than the more slowly moving stream.

In the injector, the action of a jet of steam carrying entrained water, when it meets the more slowly moving water from the boiler, is quite analogous to the above simple case. Assume that a mass m_1 of steam enters the steam-nozzle with a speed s_1, and that in the combining-tube this steam is mixed with a mass m_2 of feed-water having a speed s_2. Then, the speed s_3 of the resulting mixture is given by the equation

$$m_1 s_1 + m_2 s_2 = (m_1 + m_2) s_3. \qquad (135)$$

If s_3 is greater than the speed, s, with which the water in the boiler would discharge backward through the empty injector, the mass $(m_1 + m_2)$ will enter the boiler: otherwise, it will escape through the overflow *O*, (Fig. 109).

As it is difficult to measure directly the speeds involved in (135), it will be advantageous to replace s_1, s_2, and s_3 by their values in terms of the pressures producing them. It will be convenient to express these pressures in terms of the heights of columns of water that will exert the same pressures. Denote the pressure heads corresponding to the pressures which produce the speeds s_1, s_2, and s_3 by h_1, h_2, and h_3, respectively. Then, substituting in (135) the values of the various speeds as obtained from (127), we obtain

$$m_1 \sqrt{(2\,gh_1)} + m_2 \sqrt{(2\,gh_2)} = (m_1 + m_2) \sqrt{(2\,gh_3)},$$

or

$$m_1 \sqrt{h_1} + m_2 \sqrt{h_2} = (m_1 + m_2) \sqrt{h_3}. \qquad (136)$$

By solving this equation for $m_2 \div m_1$ we can obtain the mass of water that a perfect injector would force into a boiler per unit mass of steam consumed.

The energy required to force this water into the boiler is due to the transformation, in the combining-tube, of a certain amount of the heat energy possessed by the entering steam into an equivalent amount of kinetic energy. The combining-tube is designed so that the entering steam shall expand in the direction of its length. This sudden expansion is the cause of the great value of s_1.

CHAPTER XIII.

PROPERTIES OF MATTER.

147. General Considerations.—In the present state of our knowledge, the laws of such properties of matter as elasticity, friction, surface tension, and solution, appear to be generalizations of isolated groups of phenomena which cannot be deduced from the five fundamental principles of dynamics but which can be derived only from the experimental study of matter under the special conditions imposed by the nature of the investigation. Generalizations based upon facts or relations obtained solely from experiment or observation are called *empirical laws.* In the present chapter several such laws will be studied.

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§1. *Elasticity.*

148. Definitions.--A force may either deform a body or change its motion. When a deformation is produced, a state of equilibrium may finally be attained between the applied force and the resistance to deformation, or force of restitution, developed in the body by the applied force. *Elasticity* is the name given to "that property in virtue of which a body requires force to change its bulk or shape, and requires a continued application of the force to maintain the change, and springs back when the force is removed; and if left without the force, does not remain at rest except in its previous bulk and shape." (Kelvin.)

A body is highly elastic which not only offers a great resistance to distortion but which also completely recovers its size and shape on the removal of the deforming force, *e. g.* steel, glass. A body is slightly elastic which is either deformed by a small force or which can sustain but a small deforming force without permanent deformation, *e. g.* rubber, clay. When a body is perfectly elastic, a deforming force will develop in it an equal and opposite force of restitution which will not diminish with the lapse of time. A body is imperfectly elastic when it does not completely recover its original shape or size on the removal of the deforming force. It is said to have then received a *permanent set* or to have been deformed beyond its *limit of elasticity.* When a body can be deformed through wide limits without receiving a permanent set,it is said to possess great *elastic toughness*; *e. g.* rubber has great toughness though it is not highly elastic. Glass, on the other hand, has little elastic toughness but is very elastic. That is, glass cannot be distorted through a considerable range, but a distortion of glass of unit amount develops a great force of restitution.

The force of restitution, per unit area, developed by a given distortion is called *stress. Strain* is the deformation produced per unit length, per unit volume, etc., depending upon the nature of the deformation.

149. Coefficient of Elasticity.—It is found that when a body is distorted within the limits of elasticity, the distortion is proportional to the distorting force. This is called Hooke's Law. It is also found that when the limits of elasticity are not exceeded, the force of restitution developed by a distortion is equal and opposite to the distorting force. It follows that so long as the distortion does not exceed a certain value peculiar to each substance, the stress varies directly with the strain. The constant ratio of the stress developed in any substance to the strain producing it, is called the *coefficient of elasticity* of the given material.

Since forces applied to a body in different ways produce different types of deformation, there are various coefficients of elasticity, as the Tensile Coefficient of Elasticity, the Bulk Coefficient of Elasticity, the Torsional Coefficient of Elasticity, etc.

150. Young's Modulus.—When a body is stretched so as not to exceed its limit of elasticity, the constant of proportionality between stress and strain is called the tensile coefficient of elasticity or *Young's modulus* of elasticity of the substance.

If l be the length of a wire, A its area of cross section, e

the elongation produced by a force F, then the Young's modulus of elasticity of the material composing the wire is

$$E = \frac{F}{A} \div \frac{e}{l} = \frac{Fl}{Ae}. \tag{137}$$

If the force be measured in dynes, and the other quantities in centimeters, then the value of E will be in dynes per square centimeter.

151. The Bulk Modulus.—When a body is compressed on all sides, without changing its shape, by an amount not to exceed its limit of elasticity, the constant of proportionality between stress and strain is called the bulk coefficient of elasticity or the *bulk modulus* of elasticity of the substance.

Let the force per unit area applied to a body of volume V be denoted by P. Then, if under the action of this force the body diminishes in volume by an amount ΔV, the bulk modulus of the material is

$$k = P \div \frac{\Delta V}{V} = \frac{P\,V}{\Delta V}. \tag{138}$$

152. Simple Rigidity.—If a rectangular parallelopiped of rubber ac has two opposite faces glued to two boards, and if one of these boards is pushed sidewise in its own plane, there is no change in the volume of the block but its shape is changed to $fgcd$. In this case the strain is the ratio of af to ad and is called a *shear* or a *shearing strain.* If F is the force applied, and A is the area of the face ab, then F divided by A is called a *shearing stress.* If the block of rubber is very thin in the direction normal to the paper, and if it is bent around until ad coincides with bc, it is seen that a shear is the kind of strain involved in the twisting of a wire about its geometrical axis. The ratio of a shearing stress to the shearing strain which it produces is called the *simple rigidity* or the *slide modulus* of the material sheared.

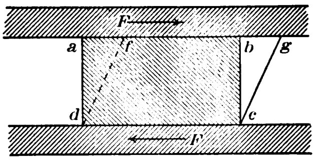

Fig. 110.

153. Vibration of Elastic Bodies.—When a body is distorted through such a small range that it is not permanently set, and then liberated, nearly all of the work done upon it is restored. The potential energy of the body due to its strained condition assumes the kinetic form, and the body, passing rapidly through its initial form and volume, suffers a distortion in the opposite sense. This develops a force of restitution in the direction opposite to the original force of restitution, thereby causing the body to go again through its mean condition. Thus the body is set into vibration.

From Hooke's Law, the force of restitution acting upon a body at any instant is directly proportional to the distortion of the body at that instant. Whence, the acceleration of any point of the body at the given instant is directly proportional to its displacement at that instant from its position of equilibrium. Consequently the vibrations of an elastic body are simple harmonic. The velocity, acceleration, and period of vibration of any particle of a vibrating elastic body can be obtained by means of the equations derived in Arts. 112 and 115.

154. Viscosity.—In the distortion of any body there is usually a sliding of some portions of matter with respect to the other portions composing the body. This may develop an internal frictional resistance which retards the relative motion of the parts of the body. That property of bodies by virtue of which they resist an instantaneous distortion is called *viscosity.* A body that has little viscosity is said to be *mobile.* Thus pitch, tar, and molasses are viscous fluids, while water, chloroform, and ether are mobile. The ratio of shearing stress to shearing strain produced in unit time is called the *coefficient of viscosity* of the substance.

A knowledge of the laws of viscosity is of importance to the engineer on account of the influence of viscosity upon the lubricating quality of oils. The coefficient of viscosity of oils varies greatly with changes of temperature. In general, the coefficient of viscosity decreases when the temperature is raised.

A well lubricated journal does not come into contact with its bearing but is kept separated from it by a film of the lubricant. In this way the friction between two metallic rubbing surfaces is replaced by the smaller viscous resistance offered by the lubricating oil. The resistance to the motion of a journal in a lubricated bearing depends upon the coefficient of viscosity of the oil used, the thickness and area of the lubricating film, and the relative speed of the two surfaces. For a lightly loaded bearing an oil should be used whose coefficient of viscosity is small. Such an oil, however, could not be used on a heavily loaded bearing because it would be squeezed out from between the moving surfaces, thus causing the journal to run dry. For a rough bearing or journal it would also be necessary to use an oil having a high coefficient of viscosity. Otherwise the rugosities of one of the metal surfaces would protrude through the lubricating film and come into direct contact with the other metal surface. Since, in general, the coefficient of viscosity of oils greatly diminishes as the temperature is raised, oils intended for use in steam cylinders must have a high coefficient of viscosity at ordinary temperature. For any particular service, a lubricating oil should be used that has the smallest coefficient of viscosity consistent with the requirement that it shall keep an unbroken film between the journal and bearing.

PROBLEMS.

1.—Taking Young's modulus for iron as 1,900,000,000,000 dynes sq. cm. find the increase in length of an iron wire 3 m. long and 1 mm. diameter, when stretched by a force of 5 kg. wt. [0.0985 cm.]

2.—Find the elongation produced in a 6 ft. iron tie rod, 0.5 in. in diameter, when subjected to a force of 500 lb. wt. Assume Young's modulus to be 28,000,000 lb. wt. per sq. in. [0.0065 in.]

3.—A 4 ft cylindrical copper rod 0.5 in. in diameter, and a 3 ft. cylindrical wrought iron rod of 0.7 in. in diameter, are to be stretched to the same amount. Compare the forces necessary to do this, and also the amounts of work expended in the two cases. Young's modulus for copper and for wrought iron are 17,000,000 and 28,000,000 lb. wt. per sq. in., respectively. [Force for iron : Force for copper = 4.3 : 1. Work in the same ratio.]

4.—A piece of wire 3 m. long and 1 mm. in diameter was found to stretch 0.85 mm. when subjected to a force of 4.5 kg. wt. Find the diameter of a 10 ft. length of wire made of the same material which will elongate 0.5 of one per cent. when subjected to a stretching force of 25 lb. wt. [0.01487 in.]

5.—A beam 4 m. long and of negligible mass, is held perpendicular to a smooth vertical wall by means of a wrought iron rod attached to the free end of the beam and to a point on the wall 4 m. above the beam. Find the area of cross section the rod must have in order that it may not extend more than 2 mm. when a body weighing 2000 kg. is suspended from the free end of the beam. Assume the iron rod to have a Young's modulus of $19(10)^{11}$ dynes per sq. cm. [4.12 sq. cm.]

§ 2. *Universal Gravitation.*

155. Newton's Law of Gravitation.—From an analysis of the laws of motion of the planets about the sun formulated by Kepler, Newton reached the conclusion that between the sun and each of its planets there is an attractive force which is directly proportional to the masses of the two bodies and inversely proportional to the square of the distance between their centers of mass. With the insight of genius, Newton at once imagined that the same law that applies to these celestial bodies, applies to all bodies whether celestial or terrestrial. After a long series of investigations he finally enunciated the law that *every portion of matter attracts every other portion of matter, and the gravitational force between any two bodies is proportional to the product of their masses, inversely proportional to the square of the distance between their centers of mass, and independent of the kind of matter and of the intervening medium.* This very far reaching generalization is called Newton's Law of Universal Gravitation.

A force existing between two bodies that tends to move them apart is termed *positive*, while one that tends to bring them together is termed *negative*. Thus, the gravitational force existing between two bodies of masses m_1 and m_2, whose centers of mass are separated by the distance d, is

$$F = -G\frac{m_1 m_2}{d^2}, \tag{139}$$

where G is a constant.

156. *The Gravitation Constant.—The simplest method of measuring the attraction between two small bodies and of determinating the gravitation constant G,(139),is by means of the torsion balance. This consists of two equal stationary masses m_1 and m_1', (Fig. 111), together with a suspended system including a very delicate vertical fiber supporting on its lower end a thin uniform horizontal rod carrying on its two ends equal masses m_2 and m_2'. The four bodies are so arranged that

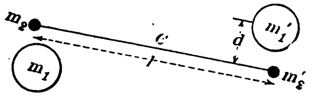

Fig. 111.

their centers of mass lie in the same horizontal plane, the distance between the centers of mass of the stationary bodies equals that between the suspended bodies, and the axis of rotation of the suspended system passes through the point midway between the centers of mass of the stationary bodies. Under these conditions the suspended system will be acted upon by a torque tending to produce rotation. If the gravitational force between m_1 and m_2 (which equals that between m_1' and m_2') when separated by a distance d be denoted by F, the torque acting on the rod is

$$Fl = -\frac{G\, m_1\, m_2\, l}{d^2}.$$

Suppose this torque twists the rod through an angle Φ, while a unit torque produces a deflection β. From Hooke's law, for the small deflection here produced, the angle through which the end of the fiber is twisted is directly proportional to the torque acting. Consequently

$$\frac{\Phi}{\beta} = \frac{G\, m_1\, m_2\, l}{d^2}.$$

Whence
$$G = \frac{\Phi\, d^2}{\beta\, m_1\, m_2\, l}.$$

This method was first used by Cavendish. Later it was considerably refined by Boys who found the gravitation constant to be

$$G = 6.6576\ (10^{-8}) \text{ in C. G. S. units.}$$

Knowing the value of the gravitation constant, together with the radius of the earth and the acceleration due to gravity, the mass and mean density of the earth can be computed. Thus, the magnitude of the gravitational force between the earth and a mass of one gram at its surface is

$$-F = 1\, g = G\, \frac{1\, m_2}{r^2}.$$

Whence
$$m_2 = \frac{g\, r^2}{G}.$$

At the equator, the radius of the earth is about 6378 kilometers, and the acceleration due to gravity is about 978 centimeters per second in one second. Using these data and assuming that the earth is a sphere, we find the mass of the earth to be about $6(10^{27})$ grams, and its mean density to be about 5.5 grams per cubic centimeter.

§ 3. *Solutions.*

157. Solution Pressure.—If some sugar be placed in water, the solid will disappear. The sugar is said to dissolve in the water and form a solution of sugar. A *solution* is defined as a homogeneous mixture of two or more substances, incapable of separation by mechanical means, in which the relative quantities of the components can vary continuously between certain limits. The *solvent* is the substance present in largest proportion and a *solute* is a substance present in less proportion. As more and more solute is added to the solution, a state may finally be reached such that, if more be introduced it will not dissolve. The solution is then said to be *saturated.* In general, more solute will dissolve in a given solvent at a high temperature than in the same solvent at a lower temperature. When a body dissolves without the occurrence of chemical action, the solution attains a lower temperature than that possessed by the solute and solvent before they were brought together. In the process of solution heat energy is always absorbed either from the surroundings or from the substances themselves.

The fact that some substances dissolve more readily in a given solvent than do others is described by the statement that every substance has a definite *solution pressure* in a given solvent. The solution pressure of a substance is the measure of its tendency to go into solution.

158. Diffusion.—When two or more miscible liquids differing in composition but which do not react chemically upon one another are placed in contact, the liquids will gradually mix until the whole mass is homogeneous. This operation takes place even though the upper layers are much less dense than the lower layers. This operation is called *diffusion.* Gases diffuse more rapidly than do liquids. Even solids diffuse when brought into intimate contact, although in this case the diffusion is very slow. This fact suggests that matter is composed of discrete particles which are in motion relative to one another.

159. Osmotic Pressure.—The force per unit area with which a substance tends to diffuse throughout a solvent and thus occupy more space is called *osmotic pressure.* Osmotic pressure can be measured by means of a membrane that will permit the passage of the solvent but will prevent the passage of the solute. Such a membrane is said to be semi-permeable. Let a solution of some substance, sugar in water, for example, be placed in a cell (*A*, Fig. 112) containing an aperture covered by a membrane that will permit the passage of water but will prevent that of the dissolved sugar. If this cell be placed in a jar of water, water will pass through the semi-permeable membrane in both directions. Acting on the inside of the membrane are two pressures—one due to the tendency of the water and another due to the tendency of the sugar to diffuse through the membrane and pass out of the cell. On the outside of the membrane there is a pressure due to the tendency of water to diffuse through the membrane into the cell. The whole of the outside surface is in contact with pure water while but a part of the inside surface is in contact with water. The remainder of the inside surface of the membrane is in contact with sugar. Since only water actually in contact with the membrane can diffuse through it, and since there is a greater area in contact with water on the outside surface than on the inside surface, it follows that more water will diffuse into the cell than will diffuse out of it. Since not any of the sugar can pass through the membrane there will be produced a pressure that can be measured by a gage or manometer.

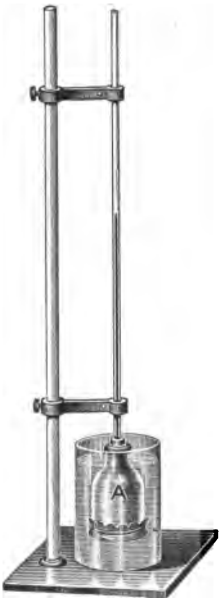

Fig. 112.

This process will continue until the total pressure on the inside of the membrane due to *water* in the cell, equals the total pressure on the outside of the membrane due to the water in the jar containing the cell. The difference between the pressures on the two sides of the membrane is then due to the sugar alone. It measures the tendency of the sugar to diffuse; in other words, it is the osmotic pressure of sugar in water at the given concentration and temperature.

On account of its solution pressure a dissolving substance goes into solution. When it is dissolved it has a certain osmotic pressure which causes it to diffuse throughout the solvent and to press upon the part which is not yet dissolved. This osmotic pressure of the part that has dissolved acts in opposition to the solution pressure of the part that is not yet dissolved. As more of the substance dissolves the osmotic pressure of the dissolved part increases until when a certain amount has dissolved, the osmotic pressure equals the solution pressure. The solution is now saturated, that is, if any more substance dissolves an equal amount will go out of solution.

§ 4. *Surface Tension.*

160. Cohesion.—If a horizontal glass plate be brought into contact with the free surface of a vessel of water, a considerable force will be required to lift the plate from the liquid surface. After the plate has been withdrawn, its surface will be found to be covered with a film of water. This shows that there is an attractive force between the particles of liquid and also an attractive force between the liquid and the glass plate. The force exerted in tearing the film of water adhering to the plate from the water remaining in the dish is a measure of the attractive force between the particles of water. The force required to separate two films of liquid, per unit area of film, is called the *cohesion* of the given liquid.

The force required to separate a film of liquid from the surface of a solid body, per unit area of surface, is called the *adhesion* of one substance to the other.

161. Surface Tension.—It is a matter of common observation that a small body of any liquid, not in contact with other bodies, assumes a globular form. This fact is explained by the mutual attraction of the particles of which the drop is conceived to be composed. A particle of liquid inside the bounding surface is acted upon by a system of forces due to the attractions of other particles on all sides of it. A particle well inside the bounding surface would be attracted no more in one direction than in another and would therefore be in equilibrium under the action of these forces. But since a particle of liquid located in the surface is not entirely surrounded by similar particles, the resultant of the attractions of the neighboring particles will urge it toward the interior of the body. If no outside force acts upon the liquid it can be shown that it will assume the shape having the least surface, viz. a spherical form. The tendency of the surface of a liquid to assume the smallest area consistent with the volume of the liquid and the external forces acting upon it, is similar to the action that would be produced if the surface of the liquid were transformed into a thin contractile membrane. The force in a liquid surface causing it to contract is called *surface tension.*

One important difference between the action of surface tension and that of a thin contractile membrane enclosing the fluid is that, whereas the tension in the membrane depends on the amount of stretching and may be greater in one direction than in another, the surface tension of a liquid is constant however much the surface is extended and at any point the surface tension is the same in all directions. With this exception. there is a close analogy between surface tension and a stretched film. This analogy will serve a useful purpose in helping one to foresee the effect of surface tension in special cases.

162. The magnitude of surface tension is measured by the force acting in the fluid surface normal to a line of unit length located in the surface. The magnitude of the tension in a given fluid surface depends upon the temperature and also upon the substance in contact with the surface. When the temperature is raised the surface tension diminishes. At 20° C., (68° F.), the tension of the surface separating water from air is 81 dynes per centimeter. At this same temperature, the tension of the surface separating olive oil from air is 37 dynes per centimeter while the tension of the surface separating olive oil from water is 20.6 dynes per centimeter. When the temperature is raised the tension in all these surfaces is diminished, although that in the surface separating air from water diminishes much more rapidly than the others.

In Fig. 113 let *A* represent a drop of olive oil placed on a surface of water. At the line of contact of air, oil, and water, there are three tensions, that in the surface separating water from air, T_1, that in the surface separating oil from air, T_2, and that in the surface separating oil from water, T_3. At the ordinary temperatures, T_1 is greater than the sum of T_2 and T_3. Consequently the oil will spread out indefinitely on the surface of the water. If the temperature of the water is raised, T_1 will diminish in value faster than T_2 and T_3, with the result that equilibrium will be attained between the tensions in the surfaces. When this occurs the drop will cease to spread. If the temperature is raised still higher, the drop will contract into a compact lenticular shape.

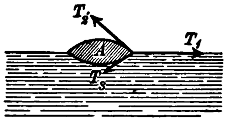

Fig. 113.

Owing to the reduction of surface tension by an increase of temperature, the oil film on a heated bearing is drawn away from the rubbing surface toward the surrounding cooler parts. This may cause the lubrication to become more and more defective until finally a "hot box" is produced.

163. If a small object be coated with a thin film of oil and be then carefully placed on the surface of water, it will float even though its density be considerably greater than water. A consideration of the forces acting on the body will render the reason obvious. As in the case considered in the foregoing article, at the line of contact of air, oil, and water there are three tensions T_1, T_2, and T_3. In addition, the small body, *B*, is acted upon by its weight downward and a force upward equal to the weight of water displaced. When the weight of the water displaced, plus the sum of the vertical components of T_1 and T_2 equals the sum of the weight of the body and the vertical component of T_3, the body will float. In this manner insects derive their support when they walk on the surface of water.

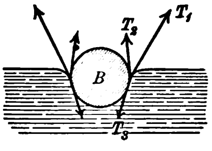

Fig. 114.

164. Capillarity.—The tension of a liquid surface in contact with a solid is very marked when the latter is a capillary tube. As this case has been considerably studied, the phenomenon of surface tension is often termed *capillarity.*

Consider a liquid in contact with a vertical solid surface. At the line of contact of air, liquid, and solid there are three tensions—that in the surface separating air and solid, T_1, that in the surface separating liquid and solid, T_2, and that in the surface separating air and liquid, T_3. If T_1 is greater than T_2, the liquid in this region will be drawn up above the level of the general surface (Fig. 115).Water and glass is an example of this case. If, however, T_2 is greater than T_1, the liquid in the region will be depressed below the level of the general surface, (Fig. 116). Mercury and glass is an example of this case.

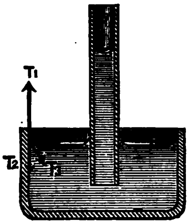

Fig. 115.

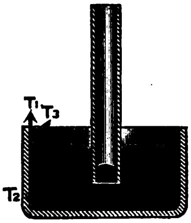

Fig. 116.

If the solid be of the form of a capillary tube, these effects will be greatly magnified. Due to surface tension, water will rise in a fine glass tube several inches above the general surface, and mercury will be correspondingly depressed below the level of the general surface.

The action of wicks in feeding oil to a flame is an example of capillarity. Again, the pores or minute spaces between particles of soil act as small tubes in bringing moisture to the surface of the ground. Anything that will enlarge these spaces will serve to diminish the capillary action. For this reason cultivation retards the evaporation of moisture from the ground.

§ 5. *Properties of Gases.*

165. Boyle's Law.—About 250 years ago, it was discovered by Robert Boyle that, to a high degree of approximation, the volume of a given mass of gas varies inversely with the pressure, so long as its temperature remains unchanged. Thus, for a given mass of gas at constant temperature

$$V \propto \frac{1}{P}.$$

Also, for a gas at given pressure and temperature,

$$V \propto m.$$

Therefore, for a gas at constant temperature,

$$V \propto \frac{m}{P}.$$

Whence, if the constant of proportionality be denoted by k, we shall have

$$V = \frac{mk}{P}.$$

or

$$PV = mk, \qquad (140)$$

where k is a constant for a particular kind of gas at a definite temperature. Thus, if at the pressures P_1, P_2, P_3, etc., a given mass of gas has the volumes V_1, V_2, V_3, etc., respectively, while the temperature remains constant, then

$$P_1V_1 = P_2V_2 = P_3V_3 = \text{etc.} = \text{a constant quantity.}$$

If the density of a gas be denoted by D, then, from (140)

$$P = kD. \qquad (141)$$

This equation, expressing Boyle's law for gases, is approximately true for most gases throughout a considerable range of pressure. A *perfect* or *ideal* gas is a hypothetical substance that would obey Boyle's law for all pressures.

In order to keep the temperature of a gas constant, heat must be added when the gas expands, and heat must be withdrawn when it is compressed. If a mass of gas of volume V_1 and pressure P_1 expands, at constant temperature, to a volume V_2 at a pressure P_2, then $P_1V_1 = P_2V_2$. But if no heat were added, the volume in the second condition, V_2', at the pressure P_2 would be less than V_2. That is, if no heat were added, P_2V_2' would be less than P_2V_2, and consequently P_1V_1 would be greater than P_2V_2'.

It is a matter of common observation that the gas jets in the upper part of a tall building burn more brightly than do those in the lower stories. Does this mean that the gas pressure is greater on the upper floors, and that it would be to the advantage of the consumer to have his meter placed in the attic instead of in the cellar?

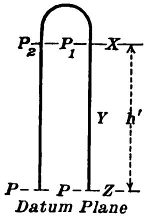

Fig. 117.

Consider a tube XYZ filled with gas and open to the air at Z. The pressure, P, of the gas at the open end equals that of the atmosphere at the same level. Let the pressure of the gas in the tube at the level of X be denoted by P_1, and the pressure of the air at the same level by P_2. Denote the mean density of the gas from X to Z by D_g, and the mean density of the air between the same levels by D_a.

Then, from (112),

$$P_1 = P - D_g gh'$$

and

$$P_2 = P - D_a gh'.$$

Whence,

$$P_1 - P_2 = gh'(D_a - D_g).$$

Consequently, if the density of the gas is less than that of air, the pressure inside the tube is always greater than the pressure outside at the same level, and this difference of pressure will be greater for points at greater elevations than for points at lower elevations. That is, the pressure of the gas at any point in the tube diminishes as the elevation is increased, but the difference between the pressure of gas inside the tube and that of the air outside of the tube at the same level increases as the elevation is increased. Since, then, the gas escaping from an orifice at a considerable elevation has not greater pressure nor density than the gas which escapes from an orifice at a less elevation, the greater mass of gas which escapes at the higher orifice must be due to a greater speed of efflux.

This can be shown from a consideration of (133) and (140). If the gas should flow so slowly through the orifice that the temperature remains constant, then the product of the pressure and the volume of a given mass of gas before efflux would be the same as this produce after efflux (140). And since the elevation of the gas before efflux is the same as after efflux, then in (133), $h_1 = h_2$. Whence, this equation becomes

$$e_2{}^2 = \frac{2}{m}(e_1 - e_2).$$

And since $(e_1 - e_2)$ increases when $(P_1 - P_2)$ increases (Art. 145), and $(P_1 - P_2)$ increases with the elevation of the point of efflux, it follows that the speed of efflux is greater, the greater the elevation of the orifice.

Again, if the speed of efflux is so great that the gas does not absorb heat fast enough to keep its temperature constant, then the product of the pressure and the volume of a given mass of gas before efflux is greater than the product of the pressure and volume of the same gas after the expansion. And the greater the difference between the pressures before and after expansion, the greater the difference between these products. Consequently, in this case also, the speed of efflux increases when the elevation of the orifice increases.

166. The Closed Manometer.—Boyle's law furnishes a simple method of determining the pressure of a fluid whose pressure is too great to be conveniently measured by means of an open manometer (Art. 133). The simplest form of *closed manometer* consists of a glass tube closed at one end and graduated throughout its length so as to read volumes. The open end is connected to the reservoir containing the fluid whose pressure is sought. In the tube is a column of mercury which serves to separate a mass of air, or other gas, from the fluid in the reservoir. If under atmospheric pressure P, the volume of the mass of gas in the tube be denoted by V, and its volume when subjected to the unknown pressure P be denoted by V', then from Boyle's law,

$$PV = P'V'.$$

Whence

$$P' = \frac{PV}{V'}. \tag{142}$$

This instrument is in common use by ship's captains for making soundings as a vessel approaches land. In this case there is used a narrow straight tube of uniform bore closed at one end. The inside of the tube is coated with a substance that changes color when wet with salt water. On lowering this tube into the sea, water is forced into the tube by the pressure of the water above. By measuring the length of the tube, and the distance from the end of the coloration to the closed end of the tube, the pressure of the water at the bottom of the sea can be obtained by means of Boyle's law. Knowing this pressure, and the density of water, the depth can be computed. In practice these tubes are calibrated so as to indicate depths directly, without computation.

167. Dalton's Law.—If a number of different gases which do not react upon one another are placed in the same reservoir, a homogeneous mixture is quickly formed. It has been shown from measurements made by Dalton that the pressure at any point in the mixture is equal to the sum of the pressures which each of the gases would separately exert in the given space. These facts suggest that a gas is composed of discrete particles which are in rapid motion and whose size is small compared to their distance apart.

PROBLEMS.

1.—A piston is situated in the middle of a closed cylinder 10 in. long, and there are equal masses of air in the two halves of the cylinder. The piston is pushed until it is within 1 in. of one of the ends. Compare the pressures on the two sides. [9:1.]

2.—The volume of an air-bubble increases ten-fold in rising from the bottom of a lake to its surface. If the height of the barometer is 30 in., what is the depth of the lake? [306 ft.]

3.—A barometer which has not been completely freed from air reads 76 cm. when the correct barometric height is 77 cm., the free internal height of the top of the barometer tube above the surface of the mercury in the cistern being 85 cm. Show that when this defective barometer reads 75 cm. a true barometer would stand at 75.9 cm.

4.—An empty bottle is floated on water. Into it water is gradually poured until it just begins to sink, and this happens when it is one-third full of water. The bottle is now emptied, immersed bottom upwards in the water, and gradually pushed down in that position. Taking the height of the water barometer as 34 ft. and neglecting the thickness of the bottle, show that it will just float freely when at such a depth that the level of the water inside it is 17 ft. below the surface, that above this depth it tends to rise, and that if it is lowered farther it will sink.

5.—The illuminating power of a gas flame is proportional to the mass of gas burned per hour. A certain gas works supplies at a pressure above that of the atmosphere equal to the pressure of a column of water 2 in. high. When the barometer stands at 30 in. an 18 candle power gas jet burns 5 cu. ft. per hr. If the gas costs $1.00 per 1000 cu. ft., what will be the cost of an 18 candle power light for 10 hr. when the barometer is at 28 in.? The temperature of the gas is the same in both cases. [0.536 cts.]

6.—At a place where the barometric height is 30 in., a uniform glass tube 18 in. long and closed at one end is sunk, open end down, to the bottom of the sea. When drawn up it is found that the water has wet the inside of the tube to a point 1 in. below the top. The density of mercury is 13.5 times that of sea water. Find the depth of the sea. [574 ft.]

7.—A live fish was placed in a closed glass jar full of water provided with a cylinder and piston, and a manometer. On pushing the piston downward, 5 c.c. of water was forced into the jar. This caused an increase in the volume of the contents of the jar amounting to 3.5 c.c. Therefore the air bladder of the fish was diminished in volume 1.5 c.c. The original pressure on the fish was 77 cm. of mercury, and the final pressure was 88 cm. Find the original volume of the air bladder. [12 c.c.]

8.—At place where the height of the water barometer is 33 ft. a bladder filled with air of density 0.00129 gm. per. c.c. is sunk in the sea. If the specific gravity of sea water is 1.025, find the depth to which the bladder must be sunk in order that the air in it may be equal to that of water. [26796 ft.]

MISCELLANEOUS TEST QUESTIONS IN DYNAMICS.

1.—In alighting from a moving car, why must one face in the direction the car is moving?

2.—Could a sailboat be propelled by the impact on the sail of compressed air escaping from a tank on the stern? Would it make any difference if the stream of air were directed backward over the stern?

3.—Show that there will be no component of the force of the wind tending to propel a boat when the sail is parallel either to the direction of the wind or to the keel of the boat.

4.—A wagon is longer than the platform of a set of hay scales. Show whether the correct weight can be obtained by weighing one end at a time and adding the two weights.

5.—Show why it is easier for a horse to draw a canal boat with a long rope than with a short one.

6.—If all the moving bodies on the earth, such as railroad trains, animals, etc., were to travel eastward for an indefinite length of time, what would be the ultimate effect upon the earth's motion?

7.—In helping a horse to pull a cart over an obstacle the teamster will sometimes push against the edge of the wheel. Show that this is preferable to pushing against the end of the cart.

8.—With the idea of producing an increased force of friction for a given pressure, the superintendent of a wagon factory designed a wagon with the brake-shoe at the top of the wheel. Assuming that the force of friction is greater when the speed is greater, discuss the correctness of his idea.

9.—Would it require a different force to impart a certain speed to a connon ball on the surface of the moon than it would if the ball were on the surface of the earth, (a), in a horizontal direction, (b), in the vertical direction?

10.—Could a man standing on a perfectly smooth surface either turn around or move along the surface?

11.—In the circus performance called "looping the gap," the upper portion of the circular path (Fig. 64) along which the bicycle rider passes is removed, the wheel and rider leaping across the gap and continuing the run on the other side. Show under what conditions this may be possible.

12.—Neglecting resistance offered by the air, is work done by a person walking along a horizontal road? Is work done by a sphere moving along a smooth horizontal surface in vacuum?

13.—A glass cylinder filled with water is supported by a spring balance. A mass is hung from the lower end of the spring so as to be immersed in the water. Will the spring be affected if the thread supporting the mass be burned?

14.—A corked bottle containing a live fly is placed on one pan of a delicate balance and the balance equilibrated. Show any change in the equilibrium of the balance in the following cases: (a), The fly climbs the vertical side of the bottle with uniform velocity; (b), it climbs with uniform acceleration; (c), by moving its wings it keeps itself poised in the center of the bottle; (d), a needle is stuck through its body and by means of a magnet the fly is held supported in the center of the bottle; (e), the fly falls with acceleration g.

15.—On hearing the query "which is heavier, a pound of lead or a pound of feathers," one person answers "they weigh the same," another says "the lead is heavier," a third says "the feathers are the heavier." In what sense does each of these three people use the word "heavier"?

16.—A glass of water together with a cork and a knitting needle are placed on a pan of a balance and the balance is counterpoised. If now the cork be held under the surface of the water in the glass by means of the knitting needle supported by an adjacent stand, state the force that must be supported by the stand.

17.—Give an example of the force acting upon a body being increased without the expenditure of energy.

18.—Is there any energy required to cause a body moving with constant linear speed to move in a circular path? If not, state the reason; if so, find its value and show its origin.

19.—If a globule of oil on a vertical shaft be rotated in a mixture of water and alcohol of the same density as the oil, the globule will assume a spheroidal form. Would it become spheroidal, (a), if the entire vessel rotated with the shaft about the same axis as before, (b), if the shaft were fixed and the vessel rotated about it as an axis?

20.—On account of the spheroidal shape of the earth, the mouth of the Mississippi River is farther from the center of the earth than is the source. That is, the river flows up hill. How is this possible?

21.—A man standing in the stern of a boat going 10 mi. per hr., projects a stone over the stern with an equal speed, with reference to the boat. Find the velocity of the stone relative to the earth. What becomes of the energy given the stone by the man?

22.—We are furnished with two spheres of the same mass and diameter. One is solid and one is hollow. Without injuring the spheres how can we determine which is the solid one?

23.—We are furnished with two eggs, one raw and one hard boiled. Without breaking the shells how can we determine which is the cooked one?

24.—A driving belt supplies a constant torque to a line of shafting. Explain why the angular speed of the shafting does not increase indefinitely.

25.—Explain the following seeming paradox: Under the force of gravity a clock "weight" moves downward with constant velocity. The force of gravity is thus doing work upon it and so increasing its energy. But in a lower position the "weight" has less potential energy than in a higher, and since it moves uniformly its kinetic energy is not changing. The "weight" is therefore losing energy.

26.—A mercury barometer is suspended from a spring and its lower end dips into a vessel of mercury supported on a separate stand. What is the load supported by the spring?